RAIRO-Oper. Res. 57 (2023) 3191–3222
https://doi.org/10.1051/ro/2023174

RAIRO Operations Research
www.rairo-ro.org

INTERVAL-VALUED PICTURE FUZZY AGGREGATION INFORMATION BASED ON FRANK OPERATORS AND THEIR APPLICATION IN GROUP DECISION MAKING

Mohammad Ali Abolfathi* and Ali Ebadian

Abstract. The information aggregation operator plays a key rule in the multiple attribute group decision making problems. The aim of this paper is to investigate the information aggregation operators method under the interval-valued picture fuzzy environment with the help of Frank norms operations. The picture fuzzy set is an extended version of the intuitionistic fuzzy set, which not only considers the degree of acceptance or rejection but also takes into the account of neutral degree during the analysis. Under these environments, some basic aggregation operators namely interval-valued picture fuzzy Frank weighted averaging operator, interval-valued picture fuzzy Frank ordered weighted averaging operator, interval-valued picture fuzzy Frank hybrid averaging operator, interval-valued picture fuzzy Frank weighted geometric operator, interval-valued picture fuzzy Frank ordered weighted geometric operator, and interval-valued picture fuzzy Frank hybrid geometric operator are proposed in this paper. Some general properties of these operators such as idempotency, commutativity, monotonicity, and boundedness, are discussed. Furthermore, the method for multiple attribute group decision making problems based on these operators is developed and the operational processes are illustrated in detail. Finally, an illustrative example has been given to show the decision steps of the proposed method and to demonstrate their effectiveness.

Mathematics Subject Classification. 90B50, 91A35.

Received October 16, 2021. Accepted November 4, 2023.

1. Introduction and preliminaries

Multiple Attribute Group Decision Making (MAGDM) problems are widespread in daily life decision situations. This process aims at selecting the best alternatives from a set of available alternatives with respect to multiple attributes, both qualitative and quantitative. In many real-world situations, due to the increasing complexity of the socio-economic environment, there are numerous limiting factors such as a lack of information and knowledge, uncertainty of the decision-making environment, difficulties in information extraction, etc. Therefore, it becomes difficult for decision maker to express values of attributes in a numeric fashion. Instead, it is more suitable to express preferences by fuzzy numbers. Many theoretical studies on MAGDM problems have been put forward in recent years [8, 17, 21, 31].

Keywords. Group decision making, multiple attribute decision making, interval-valued picture fuzzy numbers, Frank aggregation operators, picture fuzzy Frank aggregation operators.

Department of Mathematics, Urmia University, P.O. Box 165, Urmia, Iran.
*Corresponding author: m.abolfathi@urmia.ac.ir

Fuzzy theory is an important tool to process fuzzy information. Zadeh [35] firstly proposed the Fuzzy Sets (FSs) theory, then Atanassov [1,2] proposed the Intuitionistic Fuzzy Sets (IFSs) by adding a non-membership function. Furthermore, Atanassov and Gargov [4], and Atanassov [3] proposed the Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs) in which the membership and non-membership degrees were extended to interval numbers. Chen *et al.* [9] proposed a fuzzy ranking method for Interval-Valued Intuitionistic Fuzzy Numbers (IVIFNs) based on likelihood-based comparison relations, and then presented a new multi-attribute decision making method based on the proposed Interval-Valued Intuitionistic Weighted Average (IVIWA) operator. Wang *et al.* [26] defined a new score function for the IVIFNs based on the prospect value functions. Xu and Yager [32] developed a new similarity measure between IVIFSs and utilized it to solve the group decision making problems where attribute values are IVIFNs. Wang *et al.* [27] proposed a decision approach based on interval belief degrees and fuzzy evidential reasoning for multi-criteria decision problems in which the criteria weights are interval numbers and criteria values are triangular intuitionistic fuzzy numbers. In the past few decades, researchers have paid a great attention to these theories and have been successfully applied to many practical areas such as decision-making, pattern recognition, medical diagnosis, and clustering analysis [15,16,18].

Although, Atanassov's intuitionistic fuzzy set theory has been successfully applied in different areas, but there are situations in real life which could not be symbolized in IFS theory. For example, in the situation of voting, human opinions involving more answers of the types: yes, abstain, no, refusal, which cannot be accurately represented in IFSs. Also, if an expert takes an opinion from a certain person about the certain object, then a person may say that 0.2 is possibility that statement is true, 0.4 says that the statement is false and 0.1 says that he or she is not sure of it. This issue is also not handled by the FSs or IFSs. Cuong [10,11] covered these gaps by adding the neutral function in IFSs theory, he introduced the core idea of the Picture Fuzzy Sets (PFSs) model and the PFSs notion is the extension of IFSs model. Cuong basically added the neutral term along with the positive membership degree and negative membership degree o the IFSs theory. The only constraint is that in PFSs theory, the sum of the positive membership, neutral and negative membership degrees of the function is equal to or less than 1. At the present, some authors have investigated the problems under the PFSs environment. Son [23] presented a generalized picture distance measure and applied it to solve the clustering problem under the PFSs environment. Wei [28] presented a decision-making method based on the picture fuzzy weighted cross-entropy and unitized it to rank the different alternatives.

Thus, keeping inspiration from the fact that PFSs have the great powerful ability to model the imprecise and ambiguous information in real-world applications the main purpose of this paper is to present some aggregation operators under the interval-valued picture fuzzy information and named as interval-valued picture fuzzy aggregation operators for aggregating the different preference of the decision-makers during the decision-making process. In order to achieve it, firstly, some new operational laws on the IVPFSs have been introduced and their corresponding aggregation operators, namely interval-valued picture fuzzy weighted, ordered weighted and hybrid weighted aggregation operators, have been proposed. Some of its desirable properties have also been studied. Finally, these operators have been tested on the MAGDM problem for showing the effectiveness of it.

Definition 1.1. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, then an intuitionistic fuzzy set A in X is given by:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle, x \in X\}, \tag{1.1}$$

where $\mu_A(x) : X \to [0,1]$ and $\nu_A(x) : X \to [0,1]$ on condition that $0 \le \mu_A(x)+\nu_A(x) \le 1, \forall x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent the membership degree and non-membership degree of the element x to the set A, respectively.

Definition 1.2. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, then an picture fuzzy set A in X is given by:

$$A = \{\langle x, \mu_A(x), \xi_A(x), \nu_A(x)\rangle, x \in X\}, \tag{1.2}$$

where $\mu_A(x) : X \to [0,1]$, $\xi_A(x) : X \to [0,1]$ and $\nu_A(x) : X \to [0,1]$ on condition that $0 \le \mu_A(x) + \xi_A(x) + \nu_A(x) \le 1$, for all $x \in X$. The numbers $\mu_A(x)$, $\xi_A(x)$ and $\nu_A(x)$ represent the membership degree, neutral degree and non-membership degree of the element x to the set A, respectively.

Furthermore, for all $x \in X$, $\pi_A(x) = 1-\mu_A(x)-\xi_A(x)-\nu_A(x)$ is said to be the degree of refusal membership. It is obvious that $0 \le \pi_A(x) \le 1$, for all $x \in X$ and if $\xi_A(x) = 0$ then PFSs become IFSs while if $\xi_A(x) = \nu_A(x) = 0$ then PFSs become FSs. To give element x, the pair $(\mu_A(x), \xi_A(x), \nu_A(x))$ is called an Picture Fuzzy Value (PFV), which for convenience can be denoted as $a = (\mu_a, \xi_a, \nu_a)$ such that $\mu_a(x) \in [0,1]$, $\xi_a(x) \in [0,1]$ and $\nu_a(x) \in [0,1]$ and $0 \le \mu_a(x) + \xi_a(x) + \nu_a(x) \le 1$, for all $x \in X$.

Definition 1.3. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, then an Interval-Value Picture Fuzzy Set (IVPFS) $\widetilde{A}$ in X is given by:

$$\widetilde{A} = \Big\{ \Big\langle x, \widetilde{\mu}_{\widetilde{A}}(x), \widetilde{\xi}_{\widetilde{A}}(x), \widetilde{\nu}_{\widetilde{A}}(x) \Big\rangle, x \in X \Big\}, \tag{1.3}$$

where $\widetilde{\mu}_{\widetilde{A}}(x) \subseteq [0,1]$, $\widetilde{\xi}_{\widetilde{A}}(x) \subseteq [0,1]$ and $\widetilde{\nu}_{\widetilde{A}}(x) \subseteq [0,1]$ are interval numbers on condition that $\sup \widetilde{\mu}_{\widetilde{A}}(x) + \sup \widetilde{\xi}_{\widetilde{A}}(x) + \sup \widetilde{\nu}_{\widetilde{A}}(x) \le 1$, for all $x \in X$. The interval $\widetilde{\mu}_{\widetilde{A}}(x)$, $\widetilde{\xi}_{\widetilde{A}}(x)$ and $\widetilde{\nu}_{\widetilde{A}}(x)$ represent the membership degree and neutral degree and non-membership degree of the element x to the set $\widetilde{A}$ respectively.

Since $\widetilde{\mu}_{\widetilde{A}}(x)$, $\widetilde{\xi}_{\widetilde{A}}(x)$ and $\widetilde{\nu}_{\widetilde{A}}(x)$ for all $x \in X$ are closed intervals rather than real numbers and their lower and upper boundaries are denoted by $\widetilde{\mu}^L_{\widetilde{A}}(x)$, $\widetilde{\mu}^U_{\widetilde{A}}(x)$, $\widetilde{\xi}^L_{\widetilde{A}}(x)$, $\widetilde{\xi}^U_{\widetilde{A}}(x)$, $\widetilde{\nu}^L_{\widetilde{A}}(x)$ and $\widetilde{\nu}^U_{\widetilde{A}}(x)$ respectively. Therefore, another equivalent way to express an IVPFS $\widetilde{A}$ is

$$\widetilde{A} = \Big\{ \Big\langle x, \Big[\widetilde{\mu}^L_{\widetilde{A}}(x), \widetilde{\mu}^U_{\widetilde{A}}(x)\Big], \Big[\widetilde{\xi}^L_{\widetilde{A}}(x), \widetilde{\xi}^U_{\widetilde{A}}(x)\Big], \Big[\widetilde{\nu}^L_{\widetilde{A}}(x), \widetilde{\nu}^U_{\widetilde{A}}(x)\Big] \Big\rangle, x \in X \Big\} \tag{1.4}$$

where $0 \le \widetilde{\mu}^L_{\widetilde{A}}(x) \le \widetilde{\mu}^U_{\widetilde{A}}(x) \le 1,\ 0 \le \widetilde{\xi}^L_{\widetilde{A}}(x) \le \widetilde{\xi}^U_{\widetilde{A}}(x) \le 1,\ 0 \le \widetilde{\nu}^L_{\widetilde{A}}(x) \le \widetilde{\nu}^U_{\widetilde{A}}(x) \le 1$ and $\widetilde{\mu}^U_{\widetilde{A}}(x) + \widetilde{\xi}^U_{\widetilde{A}}(x) + \widetilde{\nu}^U_{\widetilde{A}}(x) \le 1$.

Similar to PFSs, for each element $x \in X$, we can compute its hesitation interval relative to $\widetilde{A}$ as:

$$\widetilde{\pi}_{\widetilde{A}}(x) = \Big[\widetilde{\pi}^L_{\widetilde{A}}(x), \widetilde{\pi}^U_{\widetilde{A}}(x)\Big] = \Big[1 - \widetilde{\mu}^U_{\widetilde{A}}(x) - \widetilde{\xi}^U_{\widetilde{A}}(x) - \widetilde{\nu}^U_{\widetilde{A}}(x), 1 - \widetilde{\mu}^L_{\widetilde{A}}(x) - \widetilde{\xi}^L_{\widetilde{A}}(x) - \widetilde{\nu}^L_{\widetilde{A}}(x)\Big].$$

Especially, if each of the intervals $\widetilde{\mu}_{\widetilde{A}}(x), \widetilde{\xi}_{\widetilde{A}}(x)$ and $\widetilde{\xi}_{\widetilde{A}}(x)$ contains exactly one element, *i.e.*, if for every $x \in X$

$$\widetilde{\mu}^L_{\widetilde{A}}(x) = \widetilde{\mu}^U_{\widetilde{A}}(x), \qquad \widetilde{\xi}^L_{\widetilde{A}}(x) = \widetilde{\xi}^U_{\widetilde{A}}(x), \qquad \widetilde{\nu}^L_{\widetilde{A}}(x) = \widetilde{\nu}^U_{\widetilde{A}}(x),$$

then the given IVPFS $\widetilde{A}$ is transformed to an ordinary PFS.

For convenience, if $\widetilde{\mu}_{\widetilde{A}}(x) = [a,b]$, $\widetilde{\xi}_{\widetilde{A}}(x) = [c,d]$ and $\widetilde{\nu}_{\widetilde{A}}(x) = [e,f]$, then $\widetilde{a} = \big([a,b],[c,d],[e,f]\big)$ is called an Interval-Valued Picture Fuzzy Number (IVPFN).

Definition 1.4. Let $\widetilde{a} = ([a,b],[c,d],[e,f])$ be an IVPFN, then score function $S(\widetilde{a})$, accuracy function $L(\widetilde{a})$ and $G(\widetilde{a})$ of an IVPFN $\widetilde{a}$ can be represented as follows:

$$S(\widetilde{a}) = \frac{a+b-c-d-e-f}{2}, \tag{1.5}$$

$$L(\widetilde{a}) = \frac{a+b+c+d+e+f}{2} \tag{1.6}$$

and

$$G(\widetilde{a}) = b+d+f-a-c-e. \tag{1.7}$$

Definition 1.5. If $\widetilde{a}_1 = ([a_1,b_1],[c_1,d_1],[e_1,f_1])$ and $\widetilde{a}_2 = ([a_2,b_2],[c_2,d_2],[e_2,f_2])$ are any two IVPFNs, then:

(1) if $S(\widetilde{a}_1) > S(\widetilde{a}_2)$, then, $\widetilde{a}_1 > \widetilde{a}_2$;
(2) if $S(\widetilde{a}_1) = S(\widetilde{a}_2)$, then:
  (a) if $L(\widetilde{a}_1) > L(\widetilde{a}_2)$, then, $\widetilde{a}_1 > \widetilde{a}_2$;
  (b) if $L(\widetilde{a}_1) = L(\widetilde{a}_2)$, then:
    (i) if $G(\widetilde{a}_1) > G(\widetilde{a}_2)$, then, $\widetilde{a}_2 > \widetilde{a}_1$;
    (ii) if $G(\widetilde{a}_1) = G(\widetilde{a}_2)$, then $\widetilde{a}_1 = \widetilde{a}_2$.

Remark 1.6. Definition 1.5 establishes an approach to comparing two IVPFNs by taking a prioritized sequence of score, accuracy and hesitation uncertainty index functions. When two IVPFNs are compared, this sequence follow sat logic order of examining the overall belonging degree, the level of accuracy or hesitation and the hesitation uncertainty index. The comparison process continues until the two IVPFNs are distinguished by one of the three function Definition 1.5. Once these two IVPFNs are differentiated at a certain priority level, the calculation terminates and functions at lower priority levels will not be computed.

Let us denote by L the lattice of non-empty intervals $L = \{[a,b]|0 \leq a \leq b \leq 1\}$, with the partial order $\leq_L$ defined as $[a,b] \leq_L [c,d] \Leftrightarrow a \leq c$ and $b \leq d$. The top and bottom elements are respectively $1_L = [1,1], 0_L = [0,0]$. The corresponding partial order in the PFNs representation $\leq_{L_{\mathrm{PFN}}}$ is defined as

$$\langle a_1, b_1, c_1\rangle \leq_{L_{\mathrm{PFN}}} \langle a_2, b_2, c_2\rangle \Leftrightarrow a_1 \leq a_2, \quad b_1 \geq b_2, \quad c_1 \geq c_2.$$

The top and bottom elements are respectively $1_{L_{\mathrm{PFN}}} =< 1,0,0 >, 0_{L_{\mathrm{PFN}}} =< 0,0,1 >$. In the context, we denote by the lattice of non-empty IVPFNs,

$$L_{\mathrm{IVPFN}} = \{\langle [a,b],[c,d],[e,f]\rangle | [a,b],[c,d],[e,f] \in D[0,1], b+d+f \leq 1\}$$

with the partial order $\leq_{L_{\mathrm{IVPFN}}}$ defined as

$$\begin{aligned}\langle [a_1,b_1],[c_1,d_1],[e_1,f_1]\rangle &\leq_{L_{\mathrm{IVPFN}}} \langle [a_2,b_2],[c_2,d_2],[e_2,f_2]\rangle \\ &\Leftrightarrow [a_1,b_1] \leq_L [a_2,b_2],[c_2,d_2] \leq_L [c_1,d_1],[e_2,f_2] \leq_L [e_1,f_1]\end{aligned} \tag{1.8}$$

where $D[0,1]$ be the set of all closed subintervals of $[0,1]$. The top and bottom elements are respectively $1_{L_{\mathrm{IVPFN}}} = \langle [1,1],[0,0],[0,0]\rangle, 0_{L_{\mathrm{IVPFN}}} = \langle [0,0],[0,0],[1,1]\rangle$.

Remark 1.7. If $\widetilde{a}_1 \leq_{L_{\mathrm{IVPFN}}} \widetilde{a}_2$ then $\widetilde{a}_1 \leq \widetilde{a}_2$, *i.e.*, the total order contains the usual partial order on L_{IVPFN}.

The triangular norm have been intensively studied, starting from Zadeh introduced max and min operation as a pair of triangular norm and triangular conorm. We can refer to various triangular norms and corresponding triangular conorms, such as product t-norm and probabilistic sum t-conorm [29], Einstein t-norm and t-conorm [25], Hamacher t-norm and t-conorm [19,20], Lukasiewicz t-norm and t-conorm [13] etc., are vehicles to operate on fuzzy sets.

All aforementioned aggregation operators are mainly based on the algebric operational laws of general t-norm and their dual t-conorms [13]. The most widely used t-norm and t-conorm is product and probabilistic sum [13], because this pair of triangular norm is useful to carry out computing. However, the main drawbacks of the probability triangular norm are that lack of flexibility and robustness. To overcome these drawbacks, some aggregation operators based on other t-norm and t-conorms have been developed over the last decades. Xia *et al.* [29] extended some families of aggregation operators based on Archimedean t-norm and t-conorm with intuitionistic fuzzy information. Wang and Liu [25] developed a family of intuitionistic fuzzy information aggregation operators with the help of Einstein t-norm and t-conorm. Liu [19] investigated Hamacher aggregation operators and discussed some of its desirable properties in detail.

Frank t-norm and t-conorm [14] which are the only one type triangle norm satisfying the compatibility. In fact Frank t-norm and t-conorm are an interesting generalizations of probabilistic and Lukasiewicz t-norm and t-conorm. Since the Frank t-norm and t-conorm involve the parameter, this can provide more flexibility and robustness in the process of information fusion and make it more adequate to model practical decision-making problems than other t-norm and t-conorm [5]. Calvo *et al.* [6] study the functional equations of Frank and Alsina for two classes of commutative, associative, and increasing binary operators. Yager [33] investigated the Additive Generating Function (AGF) of Frank t-norm and t-conorm, and established a framework with Frank t-norm and t-conorm in approximate reasoning. Casanovas and Torrens [7] investigated the additive generating function of Frank t-norms and established a framework with Frank t-norms in approximate reasoning. Sarkoci [22] made a comparison between the Frankt-norm and t-conorm and the Hamacher t-norm and t-conorm and proved that two different t-norm and t-conorm form the same family. Deschrijver [12] introduced the extensions operations to the lattice of closed Interval-Valued Fuzzy Set (IVFS) based on Frank t-norms and gave necessary and sufficient conditions such that these operations can form a complete algebraic structure. However, the previous reach works almost focus on the basic mathematical construct and and properties of Frank t-norm and t-conorm; the researches of its applications are very rare, especially in aggregation and decision making. To date, we have not seen any researches on aggregation operators based on Frank t-norm and t-conorm for decision-making problems. Therefore, it is meaningful to study aggregation operators based on Frank t-norm and t-conorm operations and their application in MAGDM under triangle interval picture fuzzy environment.

Definition 1.8. A function $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a t-norm, if below conditions are met:

(1) (Boundary) $T(0, 0) = 0; T(x, 1) = T(1, x) = x$ for all $x \in [0, 1]$;
(2) (Monotonicity) If $x_1 \leq y_1, x_2 \leq y_2$ then $T(x_1, x_2) \leq T(y_1, y_2)$;
(3) (Commutativity) $T(x_1, x_2) = T(x_2, x_1)$;
(4) (Associativity) $T(x_1, T(x_2, x_3)) = T(T(x_1, x_2), x_3)$.

A function T^*, dual to the t-norm T, defined by $T^*(x, y) = 1 - T(1 - x, 1 - y)$ is called as t-conorm.

Definition 1.9. Let $\widetilde{A}$ and $\widetilde{B}$ are any two IVPFSs; then the generalized intersection and union of $\widetilde{A}$ and $\widetilde{B}$ are defined as follows:

$$\begin{aligned}
\widetilde{A} \cap_{T,T^*} \widetilde{B} &= \left\{\left\langle x, T\big(\widetilde{\mu}_{\widetilde{A}}(x), \widetilde{\mu}_{\widetilde{B}}(x)\big), T^*\left(\widetilde{\xi}_{\widetilde{A}}(x), \widetilde{\xi}_{\widetilde{B}}(x)\right), T^*\big(\widetilde{\nu}_{\widetilde{A}}(x), \widetilde{\nu}_{\widetilde{B}}(x)\big)\right\rangle, x \in X\right\}, \\
\widetilde{A} \cup_{T,T^*} \widetilde{B} &= \left\{\left\langle x, T^*\big(\widetilde{\mu}_{\widetilde{A}}(x), \widetilde{\mu}_{\widetilde{B}}(x)\big), T\left(\widetilde{\xi}_{\widetilde{A}}(x), \widetilde{\xi}_{\widetilde{B}}(x)\right), T\big(\widetilde{\nu}_{\widetilde{A}}(x), \widetilde{\nu}_{\widetilde{B}}(x)\big)\right\rangle, x \in X\right\}. \qquad (1.9)
\end{aligned}$$

Definition 1.10. Let $\widetilde{a}_1 = ([a_1, b_1], [c_1, d_1], [e_1, f_1])$ and $\widetilde{a}_2 = ([a_2, b_2], [c_2, d_2], [e_2, f_2])$ be any two $IVPFNs$; then, the generalized intersection and union of $\widetilde{a}_1$ and $\widetilde{a}_2$ are defined as follows:

$$\begin{aligned}
\widetilde{a}_1 \otimes_{T,T^*} \widetilde{a}_2 &= ([T(a_1, a_2), T(b_1, b_2)], [T^*(c_1, c_2), T^*(d_1, d_2)], [T^*(e_1, e_2), T^*(f_1, f_2)]) \\
\widetilde{a}_1 \oplus_{T,T^*} \widetilde{a}_2 &= ([T^*(a_1, a_2), T^*(b_1, b_2)], [T(c_1, c_2), T(d_1, d_2)], [T(e_1, e_2), T(f_1, f_2)]). \qquad (1.10)
\end{aligned}$$

Frank product $\otimes_F$ is a t-norm and Frank sum $\oplus_F$ is t-conorm. They are defined in the following way:

$$T(x, y) = x \otimes_F y = \log_\lambda\left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right) \qquad (1.11)$$

$$T^*(x, y) = x \oplus_F y = 1 - \log_\lambda\left(1 + \frac{\left(\lambda^{1-x} - 1\right)\left(\lambda^{1-y} - 1\right)}{\lambda - 1}\right) \qquad (1.12)$$

for all $\lambda > 1$ and $x, y \in [0, 1]$.

It is pointed out that the Frank product and Frank sum exhibit the following properties [24]:

$$\begin{gathered}
(x \oplus_F y) + (x \oplus_F y) = x + y, \\
\frac{\partial(x \oplus_F y)}{\partial x} + \frac{\partial(x \oplus_F y)}{\partial x} = 1.
\end{gathered}$$

Based on limit theory, we can easily prove some interesting results [24]:

(i) If $\lambda \to 1$, then $x \otimes_F y \to xy, x \oplus_F y \to x + y - xy$ the Frank product and Frank sum are reduced to the probabilistic product and probabilistic sum, respectively.
(ii) If $\lambda \to \infty$, then $x \otimes_F y \to \max\{0, x + y - 1\}, x \oplus_F y \to \min\{x + y, 1\}$ the Frank product and Frank sum are reduced to the Lukasiewicz product and Lukasiewicz sum, respectively.

Because the interval-value picture fuzzy numbers are easier to express the fuzzy information and there is important significance to research aggregation operators based on Frank operations, and the generalized aggregation operators are a generalization of most aggregation operators such as arithmetic aggregation operators and geometric aggregation operators, how to extend the Frank operations to aggregate the interval-value picture fuzzy information based on the generalized aggregation operators is a meaningful works, which is also the focus of this paper. In the following, we will investigate interval-value picture generalized Frank aggregation operators which combine Frank aggregation operators with the generalized aggregation operators based on interval-value picture fuzzy information in order to generalize the existing aggregation operators.

In order to do so, the remainder of this paper is shown as follows. In Section 2, we define the Frank operation laws for IVPFNs. In Section 3, develop some interval-value picture fuzzy Frank arithmetic aggregation operators, such as Interval-Value Picture Fuzzy Frank Weighted Average (IVPFWA) operator, Interval-Value Picture Fuzzy Frank Ordered Weighted Average (IVPFOWA) operator, Interval-Value Picture Fuzzy Frank Hybird Average (IVPFHA) operator and we also discuss some desirable properties of these operators, such as commutativity, idempotency, monotonicity, boundedness, and some special cases in these operators. In Section 4, develop some interval-value picture fuzzy Frank geometric aggregation operators, such as Interval-Value Picture Fuzzy Frank Weighted Geometric (IVPFWG) operator, Interval-Value Picture Fuzzy Frank Ordered Weighted Geometric (IVPFOWG) operator, Interval-Value Picture Fuzzy Frank Hybird Geometric (IVPFHG) operator and we also discuss some desirable properties of these operators, such as commutativity, idempotency, monotonicity, boundedness, and some special cases in these operators. In Section 5, based on those operators introduced in Sections 3 and 4, we propose two methods for MAGDM problems in which attribute values take the form of IVPFNs. In Section 6, we give an example to illustrate the application of these methods.

2. Frank operations of interval-value picture fuzzy numbers

In this section, we will introduce Frank operations of picture fuzzy set and discuss some desirable properties of these operations.

Definition 2.1. Let $\widetilde{a} = \big([a,b],[c,d],[e,f]\big)$, $\widetilde{a}_1 = \big([a_1,b_1],[c_1,d_1],[e_1,f_1]\big)$ and $\widetilde{a}_2 = \big([a_2,b_2],[c_2,d_2],[e_2,f_2]\big)$ be three IVPFNs, $\lambda > 1$ and $\gamma > 0$, then

$$\begin{aligned}\widetilde{a}_1 \oplus_F \widetilde{a}_2 = \Bigg(&\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)(\lambda^{1-a_2}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)(\lambda^{1-b_2}-1)}{\lambda-1}\right)\right],\\ &\left[\log_\lambda\left(1+\frac{(\lambda^{c_1}-1)(\lambda^{c_2}-1)}{\lambda-1}\right), \log_\lambda\left(1+\frac{(\lambda^{d_1}-1)(\lambda^{d_2}-1)}{\lambda-1}\right)\right],\\ &\left[\log_\lambda\left(1+\frac{(\lambda^{e_1}-1)(\lambda^{e_2}-1)}{\lambda-1}\right), \log_\lambda\left(1+\frac{(\lambda^{f_1}-1)(\lambda^{f_2}-1)}{\lambda-1}\right)\right]\Bigg)\end{aligned} \tag{2.1}$$

$$\begin{aligned}\widetilde{a}_1 \otimes_F \widetilde{a}_2 = \Bigg(&\left[\log_\lambda\left(1+\frac{(\lambda^{a_1}-1)(\lambda^{a_2}-1)}{\lambda-1}\right), \log_\lambda\left(1+\frac{(\lambda^{b_1}-1)(\lambda^{b_2}-1)}{\lambda-1}\right)\right],\\ &\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-c_1}-1)(\lambda^{1-c_2}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-d_1}-1)(\lambda^{1-d_2}-1)}{\lambda-1}\right)\right],\\ &\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)(\lambda^{1-e_2}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-f_1}-1)(\lambda^{1-f_2}-1)}{\lambda-1}\right)\right]\Bigg)\end{aligned} \tag{2.2}$$

$$\widetilde{a}^{\wedge_F}\gamma=\left(\left[\log_\lambda\left(1+\frac{(\lambda^a-1)^\gamma}{\lambda-1}\right),\log_\lambda\left(1+\frac{(\lambda^b-1)^\gamma}{\lambda-1}\right)\right],\right.$$
$$\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-c}-1)^\gamma}{\lambda-1}\right),1-\log_\lambda\left(1+\frac{(\lambda^{1-d}-1)^\gamma}{\lambda-1}\right)\right],$$
$$\left.\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-e}-1)^\gamma}{\lambda-1}\right),1-\log_\lambda\left(1+\frac{(\lambda^{1-f}-1)^\gamma}{\lambda-1}\right)\right]\right) \tag{2.3}$$

$$\gamma\cdot_F\widetilde{a}=\left(\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-a}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right),1-\log_\lambda\left(1+\frac{(\lambda^{1-b}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right],\right.$$
$$\left[\log_\lambda\left(1+\frac{(\lambda^{c}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right),\log_\lambda\left(1+\frac{(\lambda^{d}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right],$$
$$\left.\left[\log_\lambda\left(1+\frac{(\lambda^{e}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right),\log_\lambda\left(1+\frac{(\lambda^{f}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right]\right). \tag{2.4}$$

Example 2.2. Let $\widetilde{a}_1=([0.22,0.31],[0.11,0.21],[0.33,0.37])$ and $\widetilde{a}_2=([0.13,0.33],[0.23,0.28],[0.10,0.15])$ be two IVPFNs. Suppose $\lambda=2$, then

$$\widetilde{a}_1\oplus_F\widetilde{a}_2=\left(\left[1-\log_2\left(1+\frac{(2^{1-0.22}-1)(2^{1-0.13}-1)}{2-1}\right),1-\log_2\left(1+\frac{(2^{1-0.31}-1)(2^{1-0.33}-1)}{2-1}\right)\right],\right.$$
$$\left[\log_2\left(1+\frac{(2^{0.11}-1)(2^{0.23}-1)}{2-1}\right),\log_2\left(1+\frac{(2^{0.21}-1)(2^{0.28}-1)}{2-1}\right)\right],$$
$$\left.\left[\log_2\left(1+\frac{(2^{0.33}-1)(2^{.10}-1)}{2-1}\right),\log_2\left(1+\frac{(2^{0.37}-1)(2^{0.15}-1)}{2-1}\right)\right]\right)$$
$$=([0.328,0.554],[0.020,0.046],[0.021,0.044])$$

and

$$\widetilde{a}_1\otimes_F\widetilde{a}_2=\left(\left[\log_2\left(1+\frac{(2^{0.22}-1)(2^{0.13}-1)}{2-1}\right),\log_2\left(1+\frac{(2^{0.31}-1)(2^{0.33}-1)}{2-1}\right)\right],\right.$$
$$\left[1-\log_2\left(1+\frac{(2^{1-0.11}-1)(2^{1-0.23}-1)}{2-1}\right),1-\log_2\left(1+\frac{(2^{1-0.21}-1)(2^{1-0.28}-1)}{2-1}\right)\right],$$
$$\left.\left[1-\log_2\left(1+\frac{(2^{1-0.33}-1)(2^{1-0.1}-1)}{2-1}\right),1-\log_2\left(1+\frac{(2^{1-0.37}-1)(2^{1-0.15}-1)}{2-1}\right)\right]\right)$$
$$=([0.022,0.082],[0.322,0.434],[0.404,0.457]). \tag{2.5}$$

Based on operational laws provided in Definition 2.1 we prove the following Theorem 2.3.

Theorem 2.3. *Let $\widetilde{a}=([a,b],[c,d],[e,f])$, $\widetilde{a}_1=([a_1,b_1],[c_1,d_1],[e_1,f_1])$ and $\widetilde{a}_2=([a_2,b_2],[c_2,d_2],[e_2,f_2])$ be IVPFNs and $\gamma,\gamma_1,\gamma_2\geq 0$, then*

(i) $\widetilde{a}_1\oplus_F\widetilde{a}_2=\widetilde{a}_2\oplus_F\widetilde{a}_1$,
(ii) $\widetilde{a}_1\otimes_F\widetilde{a}_2=\widetilde{a}_2\otimes_F\widetilde{a}_1$,
(iii) $\gamma\cdot_F(\widetilde{a}_1\oplus_F\widetilde{a}_2)=\gamma\cdot_F\widetilde{a}_1\oplus_F\gamma\cdot_F\widetilde{a}_2$,
(iv) $\gamma_1\cdot_F\widetilde{a}\oplus_F\gamma_2\cdot_F\widetilde{a}=(\gamma_1+\gamma_2)\cdot_F\widetilde{a}$,

(v) $\gamma_1 \cdot_F (\gamma_2 \cdot_F \widetilde{a}) = (\gamma_1 \gamma_2) \cdot_F \widetilde{a}$.

Proof. (i) It is obvious.

(ii) It is obvious.

(iii) By Definition 2.1, we get

$$
\begin{aligned}
\gamma \cdot_F (\widetilde{a}_1 \oplus_F \widetilde{a}_2) = &\left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a_1}-1\right)\left(\lambda^{1-a_2}-1\right)}{\lambda-1}\right)\right)}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right),\right.\right.\\
&1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_1}-1\right)\left(\lambda^{1-b_2}-1\right)}{\lambda-1}\right)\right)}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\Bigg],\\
&\left[\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{c_1}-1)(\lambda^{c_2}-1)}{\lambda-1}\right)\right)}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right),\ \log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{d_1}-1\right)\left(\lambda^{d_2}-1\right)}{\lambda-1}\right)\right)}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right]\\
&\left.\left[\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{e_1}-1)(\lambda^{e_2}-1)}{\lambda-1}\right)\right)}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right),\ \log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{f_1}-1\right)\left(\lambda^{f_2}-1\right)}{\lambda-1}\right)\right)}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right]\right)\\
= &\left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a_1}-1\right)^\gamma\left(\lambda^{1-a_2}-1\right)^\gamma}{(\lambda-1)^{2\gamma-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_1}-1\right)^\gamma\left(\lambda^{1-b_2}-1\right)^\gamma}{(\lambda-1)^{2\gamma-1}}\right)\right],\right.\\
&\left[\log_\lambda\left(1+\frac{(\lambda^{c_1}-1)^\gamma(\lambda^{c_2}-1)^\gamma}{(\lambda-1)^{2\gamma-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{d_1}-1\right)^\gamma\left(\lambda^{d_2}-1\right)^\gamma}{(\lambda-1)^{2\gamma-1}}\right)\right],\\
&\left.\left[\log_\lambda\left(1+\frac{(\lambda^{e_1}-1)^\gamma(\lambda^{e_2}-1)^\gamma}{(\lambda-1)^{2\gamma-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{f_1}-1\right)^\gamma\left(\lambda^{f_2}-1\right)^\gamma}{(\lambda-1)^{2\gamma-1}}\right)\right]\right).
\end{aligned}
\tag{2.6}
$$

On the other hand

$$
\begin{aligned}
\gamma \cdot_F \widetilde{a}_1 \oplus_F \gamma \cdot_F \widetilde{a}_2 = &\left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a_1}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_1}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right],\right.\\
&\left.\left[\log_\lambda\left(1+\frac{(\lambda^{c_1}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{d_1}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right],\ \left[\log_\lambda\left(1+\frac{(\lambda^{e_1}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{f_1}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right]\right)\\
&\oplus_F \left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a_2}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_2}-1\right)^\gamma}{\lambda-1}\right)\right],\right.\\
&\left.\left[\log_\lambda\left(1+\frac{(\lambda^{c_2}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{d_2}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right],\ \left[\log_\lambda\left(1+\frac{(\lambda^{e_2}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{f_2}-1\right)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right]\right)
\end{aligned}
$$

$$= \left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{1-a_2}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)}{\lambda-1}\right)\right.\right.$$

$$\left.1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{1-b_2}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)}{\lambda-1}\right)\right],$$

$$\left[\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{c_1}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{c_2}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)}{\lambda-1}\right),\right.$$

$$\left.\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{d_1}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{d_2}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)}{\lambda-1}\right)\right],$$

$$\left[\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{e_1}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{e_2}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)}{\lambda-1}\right),\right.$$

$$\left.\left.\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{f_1}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{f_2}-1)^\gamma}{(\lambda-1)^{\gamma-1}}\right)\right)}-1\right)}{\lambda-1}\right)\right]\right)$$

$$= \left(\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)^\gamma(\lambda^{1-a_2}-1)^\gamma}{(\lambda-1)^{2\gamma-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)^\gamma(\lambda^{1-b_2}-1)^\gamma}{(\lambda-1)^{2\gamma-1}}\right)\right],\right.$$
$$\left[\log_\lambda\left(1+\frac{(\lambda^{c_1}-1)^\gamma(\lambda^{c_2}-1)^\gamma}{(\lambda-1)^{2\gamma-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{d_1}-1)^\gamma(\lambda^{d_2}-1)^\gamma}{(\lambda-1)^{2\gamma-1}}\right)\right],$$
$$\left.\left[\log_\lambda\left(1+\frac{(\lambda^{e_1}-1)^\gamma(\lambda^{e_2}-1)^\gamma}{(\lambda-1)^{2\gamma-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{f_1}-1)^\gamma(\lambda^{f_2}-1)^\gamma}{(\lambda-1)^{2\gamma-1}}\right)\right]\right). \tag{2.7}$$

Therefore $\gamma \cdot_F (\widetilde{a}_1 \oplus_F \widetilde{a}_2) = \gamma \cdot_F \widetilde{a}_1 \oplus_F \gamma \cdot_F \widetilde{a}_2$.

(iv) By Definition 2.1, we have

$$\gamma_1 \cdot_F \widetilde{a} \oplus_F \gamma_2 \cdot_F \widetilde{a} = \left(\left[1-\log_\lambda\left(1+\frac{(\lambda^{1-a}-1)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b}-1)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right],\right.$$
$$\left.\left[\log_\lambda\left(1+\frac{(\lambda^{c}-1)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{d}-1)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right], \left[\log_\lambda\left(1+\frac{(\lambda^{e}-1)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{f}-1)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right]\right)$$

$$
\begin{aligned}
&\oplus_F \left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right],\right.\\
&\left.\left[\log_\lambda\left(1+\frac{(\lambda^{c}-1)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{d}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right],\ \left[\log_\lambda\left(1+\frac{(\lambda^{e}-1)^{\gamma_1}}{(\lambda-1)^{\gamma_2-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{f}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right]\right)\\
&=\left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)}{\lambda-1}\right),\right.\right.\\
&\left.1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)}{\lambda-1}\right)\right],\\
&\left[\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{c}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{c}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)}{\lambda-1}\right),\right.\\
&\left.\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{d}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{d}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)}{\lambda-1}\right)\right],\\
&\left[\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{e}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{e}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)}{\lambda-1}\right),\right.\\
&\left.\left.\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{f}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{f}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)}{\lambda-1}\right)\right]\right)\\
&=\left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a}-1\right)^{\gamma_1+\gamma_2}}{(\lambda-1)^{\gamma_1+\gamma_2-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_1}-1\right)^{\gamma}\left(\lambda^{1-b_2}-1\right)^{\gamma}}{(\lambda-1)^{\gamma_1+\gamma_2-1}}\right)\right],\right.\\
&\left[\log_\lambda\left(1+\frac{(\lambda^{c}-1)^{\gamma_1+\gamma_2}}{(\lambda-1)^{\gamma_1+\gamma_2-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{d}-1\right)^{\gamma_1+\gamma_2}}{(\lambda-1)^{\gamma_1+\gamma_2-1}}\right)\right],\\
&\left.\left[\log_\lambda\left(1+\frac{(\lambda^{e}-1)^{\gamma_1+\gamma_2}}{(\lambda-1)^{\gamma_1+\gamma_2-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{f}-1\right)^{\gamma_1+\gamma_2}}{(\lambda-1)^{\gamma_1+\gamma_2-1}}\right)\right]\right)=(\gamma_1+\gamma_2)\oplus_F \widetilde{a}.
\end{aligned}
\tag{2.8}
$$

(v) By Definition 2.1, we have

$$
\gamma_1 \cdot_F (\gamma_2 \cdot_F \widetilde{a}) = \gamma_1 \cdot_F \left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right],\right.
$$

$$\left[\log_\lambda\left(1+\frac{(\lambda^c-1)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right),\log_\lambda\left(1+\frac{\left(\lambda^d-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right],\ \left[\log_\lambda\left(1+\frac{(\lambda^e-1)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right),\log_\lambda\left(1+\frac{\left(\lambda^f-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right]\right)$$

$$=\left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right),\right.\right.$$

$$\left.1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b}-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right],$$

$$\left[\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^c-1)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right),\ \log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^d-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right],$$

$$\left.\left[\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^e-1)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right),\ \log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^f-1\right)^{\gamma_2}}{(\lambda-1)^{\gamma_2-1}}\right)\right)}-1\right)^{\gamma_1}}{(\lambda-1)^{\gamma_1-1}}\right)\right]\right)$$

$$=\left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a}-1\right)^{\gamma_1\gamma_2}}{(\lambda-1)^{\gamma_1\gamma_2-1}}\right),1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b}-1\right)^{\gamma_1\gamma_2}}{(\lambda-1)^{\gamma_1\gamma_2-1}}\right)\right],\right.$$
$$\left[\log_\lambda\left(1+\frac{(\lambda^c-1)^{\gamma_1\gamma_2}}{(\lambda-1)^{\gamma_1\gamma_2-1}}\right),\log_\lambda\left(1+\frac{\left(\lambda^d-1\right)^{\gamma_1\gamma_2}}{(\lambda-1)^{\gamma_1\gamma_2-1}}\right)\right],$$
$$\left.\left[\log_\lambda\left(1+\frac{(\lambda^e-1)^{\gamma_1\gamma_2}}{(\lambda-1)^{\gamma_1\gamma_2-1}}\right),\log_\lambda\left(1+\frac{\left(\lambda^f-1\right)^{\gamma_1\gamma_2}}{(\lambda-1)^{\gamma_1\gamma_2-1}}\right)\right]\right)=(\gamma_1\gamma_2)\cdot_F\widetilde{a} \tag{2.9}$$

which completes the proof of Theorem 2.3.

□

3. Interval-valued picture fuzzy Frank arithmetic aggregation operators

In this section, we introduce some interval-valued picture fuzzy arithmetic aggregation operators with the help of the Frank operations.

Definition 3.1. Let $\widetilde{a}_j=\left([a_j,b_j],[c_j,d_j],[e_j,f_j]\right)(j=1,2,\dots,n)$ be a collection of the IVPFFNs and let IVPFFWA : $\Omega^n\to\Omega$, if

$$\begin{aligned}\text{IVPFFWA}_\omega(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)&=\oplus_F{}_{j=1}^{n}(\omega_j\cdot_F\widetilde{a}_j)\\&=\omega_1\cdot_F\widetilde{a}_1\oplus_F\omega_2\cdot_F\widetilde{a}_2\oplus_F\dots\oplus_F\omega_n\cdot_F\widetilde{a}_n\end{aligned} \tag{3.1}$$

then the function IVPFFWA is called interval-valued picture fuzzy Frank weighted averaging (IVPFFWA) operator, where $\omega=(\omega_1,\omega_2,\dots,\omega_n)^T$ is the weight vector of $\widetilde{a}_j$ $(j=1,2,\dots,n)$ such that $\omega_j>0$ and

$\Sigma_{j=1}^{n}\omega_j = 1$. In particular, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the IVPFFWA operator reduces to the interval-valued picture fuzzy Frank averaging (IVPFFA) operator of dimension n, which is defined as follows:

$$\mathrm{IVPFFA}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) = \frac{1}{n} \cdot_F (\widetilde{a}_1 \oplus_F \widetilde{a}_2 \oplus_F \ldots \oplus_F \widetilde{a}_n).$$

Theorem 3.2. *Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs, then their aggregated value by IVPFFWA operator is still a IVPFFN, and*

$$\begin{aligned}
\mathrm{IVPFFWA}_\omega(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) &= \oplus_F{}_{j=1}^{n}(\omega_j \cdot_F \widetilde{a}_j) \\
&= \left(\left[1-\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-a_j}-1\right)^{w_j}\right), 1-\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-b_j}-1\right)^{w_j}\right)\right],\right. \\
&\quad \left[\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{c_j}-1)^{w_j}\right), \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{d_j}-1\right)^{w_j}\right)\right], \\
&\quad \left.\left[\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{e_j}-1)^{w_j}\right), \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{f_j}-1\right)^{w_j}\right)\right]\right)
\end{aligned} \tag{3.2}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\widetilde{a}_j$ $(j = 1, 2, \ldots, n)$ such that $\omega_j > 0$ and $\Sigma_{j=1}^{n}\omega_j = 1$.

Proof. We prove by using mathematical induction as follows.

For $n = 2$, based on operational laws of IVPFFNs, we have

$$\begin{aligned}
&\mathrm{IVPFFWA}_\omega(\widetilde{a}_1, \widetilde{a}_2) = \omega_1 \cdot_F \widetilde{a}_1 \oplus_F \omega_2 \cdot_F \widetilde{a}_2 \\
&= \left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a_1}-1\right)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_1}-1\right)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right)\right],\right. \\
&\quad \left.\left[\log_\lambda\left(1+\frac{(\lambda^{c_1}-1)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{d_1}-1\right)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right)\right], \left[\log_\lambda\left(1+\frac{(\lambda^{e_1}-1)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{f_1}-1\right)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right)\right]\right) \\
&\quad \oplus_F \left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a_2}-1\right)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right), 1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_2}-1\right)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)\right],\right. \\
&\quad \left.\left[\log_\lambda\left(1+\frac{(\lambda^{c_2}-1)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{d_2}-1\right)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)\right], \left[\log_\lambda\left(1+\frac{(\lambda^{e_2}-1)^{\omega_1}}{(\lambda-1)^{\omega_2-1}}\right), \log_\lambda\left(1+\frac{\left(\lambda^{f_2}-1\right)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)\right]\right) \\
&= \left(\left[1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a_1}-1\right)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-a_2}-1\right)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)\right)}-1\right)}{\lambda-1}\right),\right.\right. \\
&\quad \left.1-\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_1}-1\right)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_2}-1\right)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)\right)}-1\right)}{\lambda-1}\right)\right], \\
&\quad \left[\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{c_1}-1)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{c_2}-1)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)\right)}-1\right)}{\lambda-1}\right),\right.
\end{aligned}$$

$$\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{d_1}-1\right)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{d_2}-1\right)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)\right)}-1\right)}{\lambda-1}\right)\Bigg],$$

$$\Bigg[\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{e_1}-1)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{(\lambda^{e_2}-1)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)\right)}-1\right)}{\lambda-1}\right),$$

$$\log_\lambda\left(1+\frac{\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{f_1}-1\right)^{\omega_1}}{(\lambda-1)^{\omega_1-1}}\right)\right)}-1\right)\left(\lambda^{1-\left(1-\log_\lambda\left(1+\frac{\left(\lambda^{f_2}-1\right)^{\omega_2}}{(\lambda-1)^{\omega_2-1}}\right)\right)}-1\right)}{\lambda-1}\right)\Bigg]\Bigg)$$

$$\begin{aligned}&=\Big(\Big[1-\log_\lambda\Big(1+(\lambda^{1-a_1}-1)^{\omega_1}(\lambda^{1-a_2}-1)^{\omega_2}\Big),1-\log_\lambda\Big(1+\Big(\lambda^{1-b_1}-1\Big)^{\omega_1}\Big(\lambda^{1-b_2}-1\Big)^{\omega_2}\Big)\Big],\\&\quad\Big[\log_\lambda(1+(\lambda^{c_1}-1)^{\omega_1}(\lambda^{c_2}-1)^{\omega_1}),\log_\lambda\Big(1+\Big(\lambda^{d_1}-1\Big)^{\omega_1}\Big(\lambda^{d_2}-1\Big)^{\omega_1}\Big)\Big],\\&\quad\Big[\log_\lambda(1+(\lambda^{e_1}-1)^{\omega_1}(\lambda^{e_2}-1)^{\omega_1}),\log_\lambda\Big(1+\Big(\lambda^{f_1}-1\Big)^{\omega_1}\Big(\lambda^{f_2}-1\Big)^{\omega_1}\Big)\Big]\Big).\end{aligned}\tag{3.3}$$

Let equation (3.2) holds for $n=k$ *i.e.*,

$$\begin{aligned}\mathrm{IVPFFWA}_\omega(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_k)&=\oplus_F{}_{j=1}^{k}(\omega_j\cdot_F\widetilde{a}_j)\\&=\left(\left[1-\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{1-a_j}-1)^{w_j}\right),1-\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{1-b_j}-1)^{w_j}\right)\right],\right.\\&\quad\left[\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{c_j}-1)^{w_j}\right),\log_\lambda\left(1+\prod_{j=1}^{k}\left(\lambda^{d_j}-1\right)^{w_j}\right)\right],\\&\quad\left.\left[\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{e_j}-1)^{w_j}\right),\log_\lambda\left(1+\prod_{j=1}^{k}\left(\lambda^{f_j}-1\right)^{w_j}\right)\right]\right).\end{aligned}\tag{3.4}$$

For $n=k+1$, we have

$$\begin{aligned}&\mathrm{IVPFFWA}_\omega(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_{k+1})=\oplus_F{}_{j=1}^{k}(\omega_j\cdot_F\widetilde{a}_j)\oplus_F\omega_{k+1}\cdot_F\widetilde{a}_{k+1}\\&=\left(\left[1-\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{1-a_j}-1)^{w_j}\right),1-\log_\lambda\left(1+\prod_{j=1}^{k}\left(\lambda^{1-b_j}-1\right)^{w_j}\right)\right],\right.\\&\quad\left[\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{c_j}-1)^{w_j}\right),\log_\lambda\left(1+\prod_{j=1}^{k}\left(\lambda^{d_j}-1\right)^{w_j}\right)\right],\\&\quad\left.\left[\log_\lambda\left(1+\prod_{j=1}^{k}(\lambda^{e_j}-1)^{w_j}\right),\log_\lambda\left(1+\prod_{j=1}^{k}\left(\lambda^{f_j}-1\right)^{w_j}\right)\right]\right)\\&\quad\oplus_F\left(\left[1-\log_\lambda\left(\frac{1+\left(\lambda^{1-a_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right),1-\log_\lambda\left(1+\frac{\left(\lambda^{1-b_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right)\right],\right.\end{aligned}$$

$$
\begin{aligned}
&\left[\log_\lambda\left(1+\frac{(\lambda^{c_{k+1}}-1)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right),\log_\lambda\left(1+\frac{\left(\lambda^{d_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right)\right],\\
&\left[\log_\lambda\left(1+\frac{(\lambda^{e_{k+1}}-1)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right),\log_\lambda\left(1+\frac{\left(\lambda^{f_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}\right)\right]\Bigg)\\
=&\left(\left[1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k}\left(\lambda^{1-a_j}-1\right)^{w_j}\frac{\left(\lambda^{1-a_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}}{\lambda-1}\right),1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k}\left(\lambda^{1-b_j}-1\right)^{w_j}\frac{\left(\lambda^{1-b_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}}{\lambda-1}\right)\right],\right.\\
&\left[\log_\lambda\left(1+\frac{\prod_{j=1}^{k}(\lambda^{c_j}-1)^{w_j}\frac{\left(\lambda^{c_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}}{\lambda-1}\right),\log_\lambda\left(1+\frac{\prod_{j=1}^{k}\left(\lambda^{d_j}-1\right)^{w_j}\frac{\left(\lambda^{d_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}}{\lambda-1}\right)\right],\\
&\left.\left[\log_\lambda\left(1+\frac{\prod_{j=1}^{k}(\lambda^{e_j}-1)^{w_j}\frac{\left(\lambda^{e_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}}{\lambda-1}\right),\log_\lambda\left(1+\frac{\prod_{j=1}^{k}\left(\lambda^{f_j}-1\right)^{w_j}\frac{\left(\lambda^{f_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\omega_{k+1}-1}}}{\lambda-1}\right)\right]\right)\\
=&\left(\left[1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k}\left(\lambda^{1-a_j}-1\right)^{w_j}\left(\lambda^{1-a_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\Sigma_{j=1}^{k}\omega_j-1}(\lambda-1)^{\omega_{k+1}-1}(\lambda-1)}\right),1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k}\left(\lambda^{1-b_j}-1\right)^{w_j}\left(\lambda^{1-b_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\Sigma_{j=1}^{k}\omega_j-1}(\lambda-1)^{\omega_{k+1}-1}(\lambda-1)}\right)\right],\right.\\
&\left[\log_\lambda\left(1+\frac{\prod_{j=1}^{k}(\lambda^{c_j}-1)^{w_j}(\lambda^{c_{k+1}}-1)^{w_{k+1}}}{(\lambda-1)^{\Sigma_{j=1}^{k}\omega_j-1}(\lambda-1)^{\omega_{k+1}-1}(\lambda-1)}\right),\log_\lambda\left(1+\frac{\prod_{j=1}^{k}\left(\lambda^{d_j}-1\right)^{w_j}\left(\lambda^{d_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\Sigma_{j=1}^{k}\omega_j-1}(\lambda-1)^{\omega_{k+1}-1}(\lambda-1)}\right)\right],\\
&\left.\left[\log_\lambda\left(1+\frac{\prod_{j=1}^{k}(\lambda^{e_j}-1)^{w_j}(\lambda^{e_{k+1}}-1)^{w_{k+1}}}{(\lambda-1)^{\Sigma_{j=1}^{k}\omega_j-1}(\lambda-1)^{\omega_{k+1}-1}(\lambda-1)}\right),\log_\lambda\left(1+\frac{\prod_{j=1}^{k}\left(\lambda^{f_j}-1\right)^{w_j}\left(\lambda^{f_{k+1}}-1\right)^{w_{k+1}}}{(\lambda-1)^{\Sigma_{j=1}^{k}\omega_j-1}(\lambda-1)^{\omega_{k+1}-1}(\lambda-1)}\right)\right]\right)\\
=&\left(\left[1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}\left(\lambda^{1-a_j}-1\right)^{w_j}}{(\lambda-1)^{\Sigma_{j=1}^{k+1}\omega_j-1}}\right),1-\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}\left(\lambda^{1-b_j}-1\right)^{w_j}}{(\lambda-1)^{\Sigma_{j=1}^{k+1}\omega_j-1}}\right)\right],\right.\\
&\left[\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{c_j}-1)^{w_j}}{(\lambda-1)^{\Sigma_{j=1}^{k+1}\omega_j}-1}\right),\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}\left(\lambda^{d_j}-1\right)^{w_j}}{(\lambda-1)^{\Sigma_{j=1}^{k}\omega_j}-1}\right)\right],\\
&\left.\left[\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}(\lambda^{e_j}-1)^{w_j}}{(\lambda-1)^{\Sigma_{j=1}^{k+1}\omega_j}-1}\right),\log_\lambda\left(1+\frac{\prod_{j=1}^{k+1}\left(\lambda^{f_j}-1\right)^{w_j}}{(\lambda-1)^{\Sigma_{j=1}^{k+1}\omega_j-1}}\right)\right]\right)\\
=&\left(\left[1-\log_\lambda\left(1+\prod_{j=1}^{k+1}\left(\lambda^{1-a_j}-1\right)^{w_j}\right),1-\log_\lambda\left(1+\prod_{j=1}^{k+1}\left(\lambda^{1-b_j}-1\right)^{w_j}\right)\right],\right.\\
&\left[\log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{c_j}-1)^{w_j}\right),\log_\lambda\left(1+\prod_{j=1}^{k+1}\left(\lambda^{d_j}-1\right)^{w_j}\right)\right],\\
&\left.\left[\log_\lambda\left(1+\prod_{j=1}^{k+1}(\lambda^{e_j}-1)^{w_j}\right),\log_\lambda\left(1+\prod_{j=1}^{k+1}\left(\lambda^{f_j}-1\right)^{w_j}\right)\right]\right)
\end{aligned}
\tag{3.5}
$$

i.e., equation (3.2) holds for $n=k+1$. Therefore, equation (3.2) holds for all n, which completes the proof of Theorem 3.2. □

Example 3.3. Let $\widetilde{a}_1=([0.22,0.31],[0.11,0.24],[0.33,0.41])$, $\widetilde{a}_2=([0.03,0.25],[0.21,0.32],[0.15,0.21])$ and $\widetilde{a}_3=([0.13,0.33],[0.23,0.37],[0.10,0.25])$ be three IVPFNs. Suppose $\omega=(0.21,0.42,0.37)^T$ be the weight vector of $\widetilde{a}_i$ for $j=1,2,3,\ldots$ and $\lambda=2$, then

$$
\begin{aligned}
\mathrm{IVPFFWA}_\omega(\widetilde{a}_1,\widetilde{a}_2,\widetilde{a}_3)=\Big(\Big[&1-\log_2\Big(1+\left(2^{1-0.22}-1\right)^{0.21}\left(2^{1-0.03}-1\right)^{0.42}\left(2^{1-0.13}-1\right)^{0.37}\Big),\\
&1-\log_2\Big(1+\left(2^{1-0.31}-1\right)^{0.21}\left(2^{1-0.25}-1\right)^{0.42}\left(2^{1-0.33}-1\right)^{0.37}\Big)\Big],
\end{aligned}
$$

$$\begin{aligned}&\Big[\log_2\Big(1+\big(2^{0.11}-1\big)^{0.21}\big(2^{0.21}-1\big)^{0.42}\big(2^{0.23}-1\big)^{0.37}\Big),\\&\log_2\Big(1+\big(2^{0.24}-1\big)^{0.21}\big(2^{0.32}-1\big)^{0.42}\big(2^{0.37}-1\big)^{0.37}\Big)\Big],\\&\Big[\log_2\Big(1+\big(2^{0.33}-1\big)^{0.21}\big(2^{0.15}-1\big)^{0.42}\big(2^{0.10}-1\big)^{0.37}\Big),\\&\log_2\Big(1+\big(2^{0.41}-1\big)^{0.21}\big(2^{0.21}-1\big)^{0.42}\big(2^{0.25}-1\big)^{0.37}\Big)\Big]\Big)\\=&([0.108, 0.277], [0.189, 0.318], [0.153, 0.314]).\end{aligned}$$

Theorem 3.4. *Let* $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ *be a collection of the IVPFFNs and* $\lambda > 1$. *As* $\lambda \to 1$*, the IVPFFWA operator approaches the following limit*

$$\lim_{\lambda\to 1}\mathrm{IVPFFWA}_\omega(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) = \left(\left[1-\prod_{j=1}^n(1-a_j)^{w_j}, 1-\prod_{j=1}^n(1-b_j)^{w_j}\right],\right.$$
$$\left.\left[\prod_{j=1}^n(c_j)^{w_j}, \prod_{j=1}^n(d_j)^{w_j}\right], \left[\prod_{j=1}^n(e_j)^{w_j}, \prod_{j=1}^n(f_j)^{w_j}\right]\right).$$

Proof. Since the two components of the first interval and The four components of the second and third intervals have the same mathematical form, so we take first and third components as example, that is, we shows that

$$\lim_{\lambda\to 1}\left(1-\log_\lambda\left(1+\prod_{j=1}^n\big(\lambda^{1-a_j}-1\big)^{w_j}\right)\right) = 1-\prod_{j=1}^n(1-a_j)^{w_j}$$

and

$$\lim_{\lambda\to 1}\log_\lambda\left(1+\prod_{j=1}^n(\lambda^{c_j}-1)^{w_j}\right) = \prod_{j=1}^n c_j^{w_j}.$$

As $\lambda \to 1$, then $\prod_{j=1}^n(\lambda^{1-a_i}-1)^{\omega_j} \to 0$ and $\prod_{j=1}^n(\lambda^{c_i}-1)^{\omega_j} \to 0$, by logarithmic transform and the rule of equivalent infinitesimal replacement, we have

$$\log_\lambda\left(1+\prod_{j=1}^n\big(\lambda^{1-a_j}-1\big)^{w_j}\right) = \frac{\ln\Big(1+\prod_{j=1}^n\big(\lambda^{1-a_j}-1\big)^{w_j}\Big)}{\ln\lambda} \to \frac{\prod_{j=1}^n\big(\lambda^{1-a_j}-1\big)^{w_j}}{\ln\lambda}$$

and

$$\log_\lambda\left(1+\prod_{j=1}^n(\lambda^{c_j}-1)^{w_j}\right) = \frac{\ln\Big(1+\prod_{j=1}^n(\lambda^{c_j}-1)^{w_j}\Big)}{\ln\lambda} \to \frac{\prod_{j=1}^n(\lambda^{c_j}-1)^{w_j}}{\ln\lambda}.$$

Based on Taylor'expansion formula, we have

$$\lambda^{1-a_i} = 1+(1-a_i)\ln\lambda+\frac{(1-a_i)^2}{2}(\ln\lambda)^2+\ldots \tag{3.6}$$

and

$$\lambda^{c_i} = 1+c_i\ln\lambda+\frac{c_i^2}{2}(\ln\lambda)^2+\ldots \tag{3.7}$$

Also sine $\lambda > 1$, then $\ln\lambda > 0$, $\lambda^{1-a_i} = 1 + (1-a_i)\ln\lambda + \mathcal{O}(\ln\lambda)$ and $\lambda^{c_i} = 1 + c_i \ln\lambda + \mathcal{O}(\ln\lambda)$, it follows that

$$\lambda^{1-a_j} - 1 \to (1-a_j)\ln\lambda$$

and

$$\lambda^{c_j} - 1 \to c_j \ln\lambda.$$

Therefore

$$\begin{aligned}
\left(\lambda^{1-a_i}-1\right)^{\omega_j} \to ((1-a_j)\ln\lambda)^{\omega_j} &\Rightarrow \prod_{j=1}^{n}\left(\lambda^{1-a_i}-1\right)^{\omega_j} \to \prod_{j=1}^{n}(1-a_j)^{\omega_j}\prod_{j=1}^{n}(\ln\lambda)^{\omega_j}\\
&\Rightarrow \prod_{j=1}^{n}\left(\lambda^{1-a_i}-1\right)^{\omega_j} \to \prod_{j=1}^{n}(1-a_j)^{\omega_j}(\ln\lambda)^{\sum_{j=1}^{n}\omega_j}\\
&\Rightarrow \prod_{j=1}^{n}\left(\lambda^{1-a_i}-1\right)^{\omega_j} \to \prod_{j=1}^{n}(1-a_j)^{\omega_j}(\ln\lambda)\\
&\Rightarrow \frac{\prod_{j=1}^{n}\left(\lambda^{1-a_i}-1\right)^{\omega_j}}{\ln\lambda} \to \prod_{j=1}^{n}(1-a_j)^{\omega_j},
\end{aligned}$$

and

$$\begin{aligned}
(\lambda^{c_i}-1)^{\omega_j} \to (c_j\ln\lambda)^{\omega_j} &\Rightarrow \prod_{j=1}^{n}(\lambda^{c_i}-1)^{\omega_j} \to \prod_{j=1}^{n}(c_j)^{\omega_j}\prod_{j=1}^{n}(\ln\lambda)^{\omega_j}\\
&\Rightarrow \prod_{j=1}^{n}(\lambda^{c_i}-1)^{\omega_j} \to \prod_{j=1}^{n}(c_j)^{\omega_j}(\ln\lambda)^{\sum_{j=1}^{n}\omega_j}\\
&\Rightarrow \prod_{j=1}^{n}(\lambda^{c_i}-1)^{\omega_j} \to \prod_{j=1}^{n}(c_j)^{\omega_j}(\ln\lambda)\\
&\Rightarrow \frac{\prod_{j=1}^{n}(\lambda^{c_i}-1)^{\omega_j}}{\ln\lambda} \to \prod_{j=1}^{n}(c_j)^{\omega_j}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\lim_{\lambda\to1}\mathrm{IVPFFWA}_{\omega}(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n) = \lim_{\lambda\to1}\Bigg(&\left[1-\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-a_j}-1\right)^{w_j}\right), 1-\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-b_j}-1\right)^{w_j}\right)\right],\\
&\left[\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{c_j}-1)^{w_j}\right), \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{d_j}-1\right)^{w_j}\right)\right],\\
&\left[\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{e_j}-1)^{w_j}\right), \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{f_j}-1\right)^{w_j}\right)\right]\Bigg)\\
= \Bigg(&\left[1-\prod_{j=1}^{n}(1-a_j)^{w_j}, 1-\prod_{j=1}^{n}(1-b_j)^{w_j}\right], \left[\prod_{j=1}^{n}(c_j)^{w_j}, \prod_{j=1}^{n}(d_j)^{w_j}\right], \left[\prod_{j=1}^{n}(e_j)^{w_j}, \prod_{j=1}^{n}(f_j)^{w_j}\right]\Bigg) \qquad (3.8)
\end{aligned}$$

which completes the proof of Theorem 3.4. □

Theorem 3.5. *Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs and $\lambda > 1$. As $\lambda \to \infty$, the IVPFFWA operator approaches the following limit*

$$\lim_{\lambda\to\infty} \mathrm{IVPFFWA}_\omega(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) = \left(\left[\sum_{j=1}^n w_j a_j, \sum_{j=1}^n w_j b_j\right], \left[\sum_{j=1}^n w_j c_j, \sum_{j=1}^n w_j d_j\right], \left[\sum_{j=1}^n w_j e_j, \sum_{j=1}^n w_j f_j\right]\right).$$

Proof. Since the two components of the first interval and The four components of the second and third intervals have the same mathematical form, so we take first and third components as example, that is, we shows that

$$\lim_{\lambda\to\infty} 1 - \log_\lambda\left(1 + \prod_{j=1}^n \left(\lambda^{1-a_j} - 1\right)^{w_j}\right) = \sum_{j=1}^n w_j a_j$$

and

$$\lim_{\lambda\to\infty} \log_\lambda\left(1 + \prod_{j=1}^n (\lambda^{c_j} - 1)^{w_j}\right) = \sum_{j=1}^n w_j c_j.$$

By logarithmic transform and the rule of equivalent infinitesimal replacement and based on the L'Hospital's rule, we have

$$\begin{aligned}
\lim_{\lambda\to\infty}\left(1 - \log_\lambda\left(1 + \prod_{j=1}^n \left(\lambda^{1-a_j} - 1\right)^{w_j}\right)\right) &= 1 - \lim_{\lambda\to\infty} \frac{\ln\left(1 + \prod_{j=1}^n \left(\lambda^{1-a_j} - 1\right)^{w_j}\right)}{\ln \lambda} \\
&= 1 - \lim_{\lambda\to\infty} \frac{\frac{\prod_{j=1}^n \left(\lambda^{1-a_j}-1\right)^{w_j}}{1+\prod_{j=1}^n \left(\lambda^{1-a_j}-1\right)^{w_j}} \left(\sum_{j=1}^n \omega_j (1-a_j) \frac{\lambda^{-a_j}}{\lambda^{1-a_j}-1}\right)}{\frac{1}{\lambda}} \\
&= 1 - \lim_{\lambda\to\infty} \frac{\prod_{j=1}^n \left(\lambda^{1-a_j} - 1\right)^{w_j}}{1 + \prod_{j=1}^n (\lambda^{1-a_j} - 1)^{w_j}} \left(\sum_{j=1}^n \omega_j (1-a_j) \frac{\lambda^{1-a_j}}{\lambda^{1-a_j} - 1}\right) \\
&= 1 - \sum_{j=1}^n \omega_j (1 - a_j) \\
&= \sum_{j=1}^n \omega_j a_j, \qquad (3.9)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\lambda\to\infty} \log_\lambda\left(1 + \prod_{j=1}^n (\lambda^{c_j} - 1)^{w_j}\right) &= \lim_{\lambda\to\infty} \frac{\ln\left(1 + \prod_{j=1}^n (\lambda^{c_j} - 1)^{w_j}\right)}{\ln \lambda} \\
&= \lim_{\lambda\to\infty} \frac{\frac{\prod_{j=1}^n (\lambda^{c_j}-1)^{w_j}}{1+\prod_{j=1}^n (\lambda^{c_j}-1)^{w_j}} \left(\sum_{j=1}^n \omega_j (c_j) \frac{\lambda^{c_j-1}}{\lambda^{c_j}-1}\right)}{\frac{1}{\lambda}} \\
&= \lim_{\lambda\to\infty} \frac{\prod_{j=1}^n (\lambda^{c_j} - 1)^{w_j}}{1 + \prod_{j=1}^n (\lambda^{c_j} - 1)^{w_j}} \left(\sum_{j=1}^n \omega_j (c_j) \frac{\lambda^{c_j-1}}{\lambda^{c_j}}\right) \\
&= \sum_{j=1}^n \omega_j c_j. \qquad (3.10)
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{\lambda\to\infty} \mathrm{IVPFFWA}_\omega(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)\\
&= \lim_{\lambda\to 1}\left(\left[1-\log_\lambda\left(1+\prod_{j=1}^n\left(\lambda^{1-a_j}-1\right)^{w_j}\right), 1-\log_\lambda\left(1+\prod_{j=1}^n\left(\lambda^{1-b_j}-1\right)^{w_j}\right)\right],\right.\\
&\left[\log_\lambda\left(1+\prod_{j=1}^n(\lambda^{c_j}-1)^{w_j}\right), \log_\lambda\left(1+\prod_{j=1}^n\left(\lambda^{d_j}-1\right)^{w_j}\right)\right],\\
&\left.\left[\log_\lambda\left(1+\prod_{j=1}^n(\lambda^{e_j}-1)^{w_j}\right), \log_\lambda\left(1+\prod_{j=1}^n\left(\lambda^{f_j}-1\right)^{w_j}\right)\right]\right)\\
&\left(\left[\sum_{j=1}^n\omega_j a_j, \sum_{j=1}^n\omega_j b_j\right], \left[\sum_{j=1}^n\omega_j c_j, \sum_{j=1}^n\omega_j d_j\right], \left[\sum_{j=1}^n\omega_j e_j, \sum_{j=1}^n\omega_j f_j\right]\right)
\end{aligned}$$

which completes the proof of Theorem 3.5. □

Theorem 3.6. *Let $\widetilde{a}_j = ([a_j,b_j],[c_j,d_j],[e_j,f_j])(j=1,2,\ldots,n)$ be a collection of the IVPFFNs and $\lambda>1$, then the IVPFFWA operator is monotonically decreasing with respect to λ.*

Proof. Since the two components of the first interval and The four components of the second and third intervals have the same mathematical form, so we take first and third components as example. Let $F(\lambda) = 1-\log_\lambda(1+\prod_{j=1}^n(\lambda^{1-a_j}-1)^{w_j})$. We have

$$\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda} = -\frac{\prod_{j=1}^n\left(\lambda^{1-a_j}-1\right)^{\omega_j}\left(\sum_{j=1}^n\frac{\omega_j(1-a_j)}{\lambda-\lambda^{a_j}}\right)}{\left(1+\prod_{j=1}^n(\lambda^{1-a_j}-1)^{\omega_j}\right)\ln\lambda}.$$

Since $\lambda>1, \omega_j\le 1$ and $a_j\ge 0$, we obtain $\frac{\mathrm{d}F(\lambda)}{\mathrm{d}\lambda}<0$. Hence, $F(\lambda)$ is monotonically decreasing with respect to λ.

Also let $G(\lambda) = \log_\lambda\left(1+\prod_{j=1}^n(\lambda^{c_j}-1)^{w_j}\right)$. We have

$$\frac{\mathrm{d}G(\lambda)}{\mathrm{d}\lambda} = \frac{\prod_{j=1}^n(\lambda^{c_j}-1)^{\omega_j}\left(\sum_{j=1}^n\frac{\omega_j c_j}{\lambda-\lambda^{1-c_j}}\right)}{\left(1+\prod_{j=1}^n(\lambda^{c_j}-1)^{\omega_j}\right)\ln\lambda}.$$

Since $\lambda>1, \omega_j\le 1$ and $c_j\ge 0$, we obtain $\frac{\mathrm{d}G(\lambda)}{\mathrm{d}\lambda}>0$. Hence, $G(\lambda)$ is monotonically increasing with respect to λ.

Hence the IVPFFWA operator is monotonically decreasing with respect to λ. □

Theorem 3.7 (Idempotency)**.** *Let $\widetilde{a}_j = ([a_j,b_j],[c_j,d_j],[e_j,f_j])(j=1,2,\ldots,n)$ be a collection of the IVPFFNs. If all $\widetilde{a}_j(j=1,2,\ldots,n)$ are equal, that is, $\widetilde{a}_j=\widetilde{a}$ for all j, then*

$$\mathrm{IVPFFWA}_\omega(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)=\widetilde{a} \tag{3.11}$$

where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of $\widetilde{a}_j(j=1,2,\ldots,n)$ such that $\omega_j>0$ and $\Sigma_{j=1}^n\omega_j=1$.

Proof. By using Theorem 3.2, we have

$$\begin{aligned}
\mathrm{IVPFFWA}_{\omega}(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n) &= \left(\left[1-\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-a_j}-1\right)^{w_j}\right), 1-\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-b_j}-1\right)^{w_j}\right)\right],\right.\\
&\left[\log_{\lambda}\left(1+\prod_{j=1}^{n}(\lambda^{c_j}-1)^{w_j}\right), \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{d_j}-1\right)^{w_j}\right)\right],\\
&\left.\left[\log_{\lambda}\left(1+\prod_{j=1}^{n}(\lambda^{e_j}-1)^{w_j}\right), \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{f_j}-1\right)^{w_j}\right)\right]\right)\\
&= \left(\left[1-\log_{\lambda}\left(1+\left(\lambda^{1-a}-1\right)^{\sum_{j=1}^{n}w_j}\right), 1-\log_{\lambda}\left(1+\left(\lambda^{1-b}-1\right)^{\sum_{j=1}^{n}w_j}\right)\right],\right.\\
&\left[\log_{\lambda}\left(1+(\lambda^{c}-1)^{\sum_{j=1}^{n}w_j}\right), \log_{\lambda}\left(1+\left(\lambda^{d}-1\right)^{\sum_{j=1}^{n}w_j}\right)\right],\\
&\left.\left[\log_{\lambda}\left(1+(\lambda^{e}-1)^{\sum_{j=1}^{n}w_j}\right), \log_{\lambda}\left(1+\left(\lambda^{f}-1\right)^{\sum_{j=1}^{n}w_j}\right)\right]\right)\\
&= \left(\left[1-\log_{\lambda}\lambda^{1-a}, 1-\log_{\lambda}\lambda^{1-b}\right], \left[\log_{\lambda}\lambda^{c}\log_{\lambda}\lambda^{d}\right], \left[\log_{\lambda}\lambda^{e}\log_{\lambda}\lambda^{f}\right]\right)\\
&= ([a,b],[c,d],[e,f]). \qquad (3.12)
\end{aligned}$$

Thus, the proof is completed. □

Theorem 3.8 (Bounded)**.** *Let $\widetilde{a}_j = ([a_j,b_j],[c_j,d_j],[e_j,f_j])(j=1,2,\ldots,n)$ be a collection of the IVPFFNs. The IVPFFWA operator lies between the* max *and* min *operators.*

$$\widetilde{a}_{\min} \leq \mathrm{IVPFFWA}_{\omega}(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n) \leq \widetilde{a}_{\max}$$

where $\widetilde{a}_{\min} = \min(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)$ and $\widetilde{a}_{\max} = \max(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)$.

Proof. Since

$$\begin{aligned}
&a_{\widetilde{a}_{\min}} \leq a_j \leq a_{\widetilde{a}_{\max}}, \qquad b_{\widetilde{a}_{\min}} \leq b_j \leq b_{\widetilde{a}_{\max}}\\
&c_{\widetilde{a}_{\max}} \leq c_j \leq c_{\widetilde{a}_{\min}}, \qquad d_{\widetilde{a}_{\max}} \leq d_j \leq d_{\widetilde{a}_{\min}}\\
&e_{\widetilde{a}_{\max}} \leq e_j \leq e_{\widetilde{a}_{\min}}, \qquad f_{\widetilde{a}_{\max}} \leq f_j \leq f_{\widetilde{a}_{\min}}
\end{aligned}$$

for all $j=1,2,\ldots,n$, then it follows that

$$\begin{aligned}
\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\max\{a_j\}}-1\right)^{w_j}\right) &\leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-a_j}-1\right)\right)^{\omega_j} \leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\min\{a_j\}}-1\right)\right),\\
\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\max\{b_j\}}-1\right)^{w_j}\right) &\leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-b_j}-1\right)\right)^{\omega_j} \leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\min\{b_j\}}-1\right)^{w_j}\right),\\
\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{\max\{c_j\}}-1\right)^{w_j}\right) &\leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{c_j}-1\right)\right)^{\omega_j} \leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{\min\{c_j\}}-1\right)^{w_j}\right),\\
\log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{\max\{d_j\}}-1\right)^{w_j}\right) &\leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{d_j}-1\right)\right)^{\omega_j} \leq \log_{\lambda}\left(1+\prod_{j=1}^{n}\left(\lambda^{\min\{d_j\}}-1\right)^{w_j}\right),
\end{aligned}$$

$$\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\max\{e_j\}}-1\right)^{w_j}\right)\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{e_j}-1)\right)^{\omega_j}\le\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\min\{e_j\}}-1\right)^{w_j}\right),$$
$$\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\max\{f_j\}}-1\right)^{w_j}\right)\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{f_j}-1)\right)^{\omega_j}\le\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\min\{f_j\}}-1\right)^{w_j}\right).$$

Then, it follows that

$$\log_\lambda\left(1+\left(\lambda^{1-\max\{a_j\}}-1\right)^{\sum_{j=1}^n w_j}\right)\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{1-a_j}-1)\right)^{\omega_j}\le\log_\lambda\left(1+\left(\lambda^{1-\min\{a_j\}}-1\right)^{\sum_{j=1}^n w_j}\right),$$
$$\log_\lambda\left(1+\left(\lambda^{1-\max\{b_j\}}-1\right)^{\sum_{j=1}^n w_j}\right)\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{1-b_j}-1)\right)^{\omega_j}\le\log_\lambda\left(1+\left(\lambda^{1-\min\{b_j\}}-1\right)^{\sum_{j=1}^n w_j}\right),$$
$$\log_\lambda\left(1+\left(\lambda^{\max\{c_j\}}-1\right)^{\sum_{j=1}^n w_j}\right)\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{c_j}-1)\right)^{\omega_j}\le\log_\lambda\left(1+\left(\lambda^{\min\{c_j\}}-1\right)^{\sum_{j=1}^n w_j}\right),$$
$$\log_\lambda\left(1+\left(\lambda^{\max\{d_j\}}-1\right)^{\sum_{j=1}^n w_j}\right)\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{d_j}-1)\right)^{\omega_j}\le\log_\lambda\left(1+\left(\lambda^{\min\{d_j\}}-1\right)^{\sum_{j=1}^n w_j}\right),$$
$$\log_\lambda\left(1+\left(\lambda^{\max\{e_j\}}-1\right)^{\sum_{j=1}^n w_j}\right)\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{e_j}-1)\right)^{\omega_j}\le\log_\lambda\left(1+\left(\lambda^{\min\{e_j\}}-1\right)^{\sum_{j=1}^n w_j}\right),$$
$$\log_\lambda\left(1+\left(\lambda^{\max\{f_j\}}-1\right)^{\sum_{j=1}^n w_j}\right)\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{f_j}-1)\right)^{\omega_j}\le\log_\lambda\left(1+\left(\lambda^{\min\{f_j\}}-1\right)^{\sum_{j=1}^n w_j}\right).$$

Hence

$$a_{\widetilde{a}_{\min}}\le 1-\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{1-a_j}-1)\right)^{\omega_j}\le a_{\widetilde{a}_{\max}},\qquad b_{\widetilde{a}_{\min}}\le 1-\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{1-b_j}-1)\right)^{\omega_j}\le b_{\widetilde{a}_{\max}},$$
$$c_{\widetilde{a}_{\max}}\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{c_j}-1)\right)^{\omega_j}\le c_{\widetilde{a}_{\min}},\qquad d_{\widetilde{a}_{\max}}\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{d_j}-1)\right)^{\omega_j}\le d_{\widetilde{a}_{\min}},$$
$$e_{\widetilde{a}_{\max}}\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{e_j}-1)\right)^{\omega_j}\le e_{\widetilde{a}_{\min}},\qquad f_{\widetilde{a}_{\max}}\le\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{f_j}-1)\right)^{\omega_j}\le f_{\widetilde{a}_{\min}}.$$

Thus, the proof is completed. □

Theorem 3.9 (Monotonicity)**.** *Let* $(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)$ *and* $(\widetilde{a^*}_1,\widetilde{a^*}_2,\ldots,\widetilde{a^*}_n)$ *be two collection of the IVPFFNs. If* $\widetilde{a}_j\le\widetilde{a^*}_j$ *for all* $j=1,2,3,\ldots,n$ *then,*

$$\mathrm{IVPFFWA}_\omega(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)\le\mathrm{IVPFFWA}_\omega\big(\widetilde{a^*}_1,\widetilde{a^*}_2,\ldots,\widetilde{a^*}_n\big).\tag{3.13}$$

Proof. Since

$$a_{\widetilde{a}_j}\le a_{\widetilde{a^*}_j},\qquad b_{\widetilde{a}_j}\le b_{\widetilde{a^*}_j}$$

$$c_{\widetilde{a^*}_j} \le c_{\widetilde{a}_j}, \qquad d_{\widetilde{a^*}_j} \le d_{\widetilde{a}_j}$$
$$e_{\widetilde{a^*}_j} \le e_{\widetilde{a}_j}, \qquad f_{\widetilde{a^*}_j} \le f_{\widetilde{a}_j}$$

for all $j = 1, 2, \ldots, n$, then it follows that

$$\begin{aligned}
\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\max\left\{a_j^*\right\}}-1\right)^{w_j}\right) &\le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\min\{a_j\}}-1\right)\right),\\
\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\max\left\{b_j^*\right\}}-1\right)^{w_j}\right) &\le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\min\{b_j\}}-1\right)^{w_j}\right),\\
\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\max\left\{c_j^*\right\}}-1\right)^{w_j}\right) &\le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\min\{c_j\}}-1\right)^{w_j}\right),\\
\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\max\left\{d_j^*\right\}}-1\right)^{w_j}\right) &\le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\min\{d_j\}}-1\right)^{w_j}\right),\\
\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\max\left\{e_j^*\right\}}-1\right)^{w_j}\right) &\le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\min\{e_j\}}-1\right)^{w_j}\right),\\
\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\max\left\{f_j^*\right\}}-1\right)^{w_j}\right) &\le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\min\{f_j\}}-1\right)^{w_j}\right).
\end{aligned} \tag{3.14}$$

Then, it follows that

$$\begin{aligned}
1-\log_\lambda\left(1+\left(\lambda^{1-\max\{a_j\}}-1\right)^{\sum_{j=1}^{n} w_j}\right) &\le 1-\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-a_j^*}-1\right)\right)^{\omega_j},\\
1-\log_\lambda\left(1+\left(\lambda^{1-\max\{b_j\}}-1\right)^{\sum_{j=1}^{n} w_j}\right) &\le 1-\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-b_j^*}-1\right)\right)^{\omega_j},\\
\log_\lambda\left(1+\left(\lambda^{\max\left\{c_j^*\right\}}-1\right)^{\sum_{j=1}^{n} w_j}\right) &\le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{c_j}-1\right)\right)^{\omega_j},\\
\log_\lambda\left(1+\left(\lambda^{\max\left\{d_j^*\right\}}-1\right)^{\sum_{j=1}^{n} w_j}\right) &\le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{d_j}-1\right)\right)^{\omega_j},\\
\log_\lambda\left(1+\left(\lambda^{\max\left\{e_j^*\right\}}-1\right)^{\sum_{j=1}^{n} w_j}\right) &\le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{e_j}-1\right)\right)^{\omega_j},\\
\log_\lambda\left(1+\left(\lambda^{\max\left\{f_j^*\right\}}-1\right)^{\sum_{j=1}^{n} w_j}\right) &\le \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{f_j}-1\right)\right)^{\omega_j}.
\end{aligned} \tag{3.15}$$

Thus, the proof is completed. □

Theorem 3.10 (Commutativity). *Let $(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)$ and $(\widetilde{a}_1^*,\widetilde{a}_2^*,\ldots,\widetilde{a}_n^*)$ be collections of the IVPFFNs; then*

$$\mathrm{IVPFFWA}_\omega(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)=\mathrm{IVPFFWA}_\omega(\widetilde{a}_1^*,\widetilde{a}_2^*,\ldots,\widetilde{a}_n^*), \tag{3.16}$$

where $(\widetilde{a}_1^*,\widetilde{a}_2^*,\ldots,\widetilde{a}_n^*)$ is any permutation of $(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)$.

Furthermore, based on the IVPFFWA operator above, we shall develop the interval-value picture fuzzy Frank ordered weighted averaging (IVPFFOWA) operator as follows:

Definition 3.11. Let $\widetilde{a}_j=\big([a_j,b_j],[c_j,d_j],[e_j,f_j]\big)(j=1,2,\ldots,n)$ be a collection of the IVPFFNs and let $\mathrm{IVPFFWA}:\Omega^n\to\Omega$, if

$$\begin{aligned}\mathrm{IVPFFWA}_\omega(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)&=\oplus_F{}_{j=1}^{n}\big(\omega_j\cdot_F\widetilde{a}_{\sigma(j)}\big)\\&=\omega_1\cdot_F\widetilde{a}_{\sigma(1)}\oplus_F\omega_2\cdot_F\widetilde{a}_{\sigma(2)}\oplus_F\ldots\oplus_F\omega_n\cdot_F\widetilde{a}_{\sigma(n)}\end{aligned} \tag{3.17}$$

then the function IVPFFOWA is called interval-valued picture fuzzy Frank ordered weighted averaging (IVPFFOWA) operator, where $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is a permutation of $(1,2,\ldots,n)$ such that $\widetilde{a}_{\sigma(j-1)}\geq\widetilde{a}_{\sigma(j)}$ for all $j=2,3,\ldots,n$ and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of $\widetilde{a}_j$ $(j=1,2,\ldots,n)$ such that $\omega_j>0$ and $\Sigma_{j=1}^n\omega_j=1$.

Theorem 3.12. *Let $\widetilde{a}_j=\big([a_j,b_j],[c_j,d_j],[e_j,f_j]\big)(j=1,2,\ldots,n)$ be a collection of the IVPFFNs, then their aggregated value by IVPFFOWA operator is still a IVPFFN, and*

$$\begin{aligned}\mathrm{IVPFFOWA}_\omega(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)&=\oplus_F{}_{j=1}^{n}\big(\omega_j\cdot_F\widetilde{a}_{\sigma(j)}\big)\\&=\Bigg(\Bigg[1-\log_\lambda\Bigg(1+\prod_{j=1}^n\big(\lambda^{1-a_{\sigma(j)}}-1\big)^{w_j}\Bigg),1-\log_\lambda\Bigg(1+\prod_{j=1}^n\big(\lambda^{1-b_{\sigma(j)}}-1\big)^{w_j}\Bigg)\Bigg],\\&\quad\Bigg[\log_\lambda\Bigg(1+\prod_{j=1}^n\big(\lambda^{c_{\sigma(j)}}-1\big)^{w_j}\Bigg),\log_\lambda\Bigg(1+\prod_{j=1}^n\big(\lambda^{d_{\sigma(j)}}-1\big)^{w_j}\Bigg)\Bigg],\\&\quad\Bigg[\log_\lambda\Bigg(1+\prod_{j=1}^n\big(\lambda^{e_{\sigma(j)}}-1\big)^{w_j}\Bigg),\log_\lambda\Bigg(1+\prod_{j=1}^n\big(\lambda^{f_{\sigma(j)}}-1\big)^{w_j}\Bigg)\Bigg]\Bigg)\end{aligned} \tag{3.18}$$

where $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is a permutation of $(1,2,\ldots,n)$ such that $\widetilde{a}_{\sigma(j-1)}\geq\widetilde{a}_{\sigma(j)}$ for all $j=2,3,\ldots,n$ and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of $\widetilde{a}_j$ $(j=1,2,\ldots,n)$ such that $\omega_j>0$ and $\Sigma_{j=1}^n\omega_j=1$.

Example 3.13. Let $\widetilde{a}_1=([0.22,0.31],[0.11,0.24],[0.33,0.41])$, $\widetilde{a}_2=([0.03,0.25],[0.21,0.32],[0.15,0.21])$ and $\widetilde{a}_3=([0.13,0.33],[0.23,0.37],[0.10,0.25])$ be three IVPFNs. Suppose $\omega=(0.30,0.50,0.20)^T$ be the position weighted vector and $\lambda=2$. The simple steps are shown as follows:

(1) Calculating score functions of $\widetilde{a}_1,\widetilde{a}_2$ and $\widetilde{a}_3$ by Definition 1.4, we can get

$$\begin{aligned}S(\widetilde{a}_1)&=\frac{0.22+0.31-0.11-0.24-0.33-0.41}{2}=-0.279\\S(\widetilde{a}_2)&=\frac{0.03+0.25-0.21-0.32-0.15-0.21}{2}=-0.305\\S(\widetilde{a}_3)&=\frac{0.13+0.33-0.23-0.37-0.10-0.25}{2}=-0.245.\end{aligned}$$

(2) Rank the IVPFFNs $\widetilde{a}_1,\widetilde{a}_2$ and $\widetilde{a}_3$ by Definition 1.5, we can get $\widetilde{a}_3>\widetilde{a}_1>\widetilde{a}_2$.
(3) Get $\sigma(j)(j=1,2,3)$ by the ranking of $\widetilde{a}_1,\widetilde{a}_2$ and $\widetilde{a}_3$, we have $\sigma(1)=3,\sigma(2)=1,\sigma(3)=2$.

(4) Calculating IVPFFOWA $(\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3)$, we have

$$
\begin{aligned}
\text{IVPFFWA}_\omega(\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3) = &\left(\left[1 - \log_2\left(1 + \prod_{j=1}^{3}\left(2^{1-a_{\sigma(j)}} - 1\right)^{w_j}\right), 1 - \log_2\left(1 + \prod_{j=1}^{3}\left(2^{1-b_{\sigma(j)}} - 1\right)^{w_j}\right)\right],\right.\\
&\left[\log_2\left(1 + \prod_{j=1}^{3}\left(2^{c_{\sigma(j)}} - 1\right)^{w_j}\right), \log_2\left(1 + \prod_{j=1}^{3}\left(2^{d_{\sigma(j)}} - 1\right)^{w_j}\right)\right],\\
&\left.\left[\log_2\left(1 + \prod_{j=1}^{3}\left(2^{e_{\sigma(j)}} - 1\right)^{w_j}\right), \log_2\left(1 + \prod_{j=1}^{3}\left(2^{f_{\sigma(j)}} - 1\right)^{w_j}\right)\right]\right)\\
= &\left(\left[1 - \log_2\left(1 + \left(2^{1-0.13} - 1\right)^{0.30}\left(2^{1-0.22} - 1\right)^{0.50}\left(2^{1-0.03} - 1\right)^{0.20}\right),\right.\right.\\
&\left.1 - \log_2\left(1 + \left(2^{1-0.33} - 1\right)^{0.30}\left(2^{1-0.31} - 1\right)^{0.50}\left(2^{1-0.25} - 1\right)^{0.20}\right)\right],\\
&\left[\log_2\left(1 + \left(2^{0.23} - 1\right)^{0.30}\left(2^{0.11} - 1\right)^{0.50}\left(2^{0.21} - 1\right)^{0.20}\right),\right.\\
&\left.\log_2\left(1 + \left(2^{0.37} - 1\right)^{0.30}\left(2^{0.24} - 1\right)^{0.50}\left(2^{0.32} - 1\right)^{0.20}\right)\right],\\
&\left[\log_2\left(1 + \left(2^{0.10} - 1\right)^{0.30}\left(2^{.33} - 1\right)^{0.50}\left(2^{0.15} - 1\right)^{0.20}\right),\right.\\
&\left.\left.\log_2\left(1 + \left(2^{0.25} - 1\right)^{0.30}\left(2^{0.41} - 1\right)^{0.50}\left(2^{0.21} - 1\right)^{0.20}\right)\right]\right)\\
= &([0.157, 0.304], [0.156, 0.289], [0.198, 0.310]).
\end{aligned}
$$

An important characteristic of the IVPFFOWA operator is that ω_j is only decided by the j-th position in ranking from largest to smallest. Therefore, can also be called the position weighted vector. In general, the position weighted vector can be determined according to actual needs. In some cases, it can be determined by some mathematical methods. For example, the method proposed by Xu [30] is shown as follows:

$$
\omega_j = \frac{e^{\frac{(j-m_{n-1})^2}{2o_{n-1}^2}}}{\sum_{j=1}^{n-1} e^{\frac{(j-m_{n-1})^2}{2\sigma_{n-1}^2}}},
$$

where is the mean of the collection of $1, 2, 3, \ldots, n-1$, and is the standard deviation. and can be calculated b the following formulas, respectively:

$$
\begin{aligned}
m_{n-1} &= \frac{n}{2}\\
o_{n-1} &= \sqrt{\frac{1}{n-1}\sum_{j=1}^{n-1}(j - m_{n-1})^2}.
\end{aligned}
$$

Similar to the IVPFFWA operator, the IVPFFOWA operator also has the operator also has the properties of monotonicity, idempotency, and boundedness.

Theorem 3.14 (Idempotency)**.** *Let* $(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n)$ *be a collection of the IVPFFNs. If all* $\widetilde{a}_j (j = 1, 2, \ldots, n)$ *are equal, that is,* $\widetilde{a}_j = \widetilde{a}$ *for all* j*, then*

$$
\text{IVPFFOWA}_\omega(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) = \widetilde{a}. \tag{3.19}
$$

Theorem 3.15 (Bounded). *Let $(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n)$ be a collection of the IVPFFNs. The IVPFFWA operator lies between the* max *and* min *operators.*

$$\max(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) \leq \mathrm{IVPFFOWA}_\omega(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) \leq \max(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n). \tag{3.20}$$

Theorem 3.16 (Monotonicity). *Let $(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n)$ and $(\widetilde{a}^*_1, \widetilde{a}^*_2, \dots, \widetilde{a}^*_n)$ be two collection of the IVPFFNs. If $\widetilde{a}_j \leq \widetilde{a}^*_j$ for all $j = 1, 2, 3, \dots, n$ then*

$$\mathrm{IVPFFOWA}_\omega(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) \leq \mathrm{IVPFFOWA}_\omega(\widetilde{a}^*_1, \widetilde{a}^*_2, \dots, \widetilde{a}^*_n). \tag{3.21}$$

In addition, it is easy to prove that the IVPFFOWA operator has the commutativity.

Theorem 3.17 (Commutativity). *Let $(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n)$ and $(\widetilde{a}^*_1, \widetilde{a}^*_2, \dots, \widetilde{a}^*_n)$ be collections of the IVPFFNs; then*

$$\mathrm{IVPFFOWA}_\omega(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) = \mathrm{IVPFFOWA}_\omega(\widetilde{a}^*_1, \widetilde{a}^*_2, \dots, \widetilde{a}^*_n), \tag{3.22}$$

where $(\widetilde{a}^*_1, \widetilde{a}^*_2, \dots, \widetilde{a}^*_n)$ is any permutation of $(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n)$.

Corollary 3.18. *Let $(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n)$ be a collection of the IVPFFNs, then*

(1) *If $\omega = (1, 0, 0, \dots, 0)^T$ then* $\mathrm{IVPFFOWA}(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) = \max\{\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n\}$.
(2) *If $\omega = (0, 0, 0, \dots, 1)^T$ then* $\mathrm{IVPFFOWA}(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) = \min\{\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n\}$.
(3) *If $\omega_j = 1$ and $\omega_i = 0 (i \neq j)$, then* $\mathrm{IVPFFOWA}(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) = \widetilde{a}_{\sigma(j)}$ *where $\widetilde{a}_{\sigma(j)}$ is the jth largest of $\widetilde{a}_j$.*

Based on the definition of IVPFFWA and IVPFFOWA operators, we can see that the operator can only reflects the self importance of each IVPFNs and IVPFFOWA operator only emphasizes the ordered position importance of each IVPFNs. In real-world practical situations, we should both consider the two aspects at the same time. Therefore, to overcome this limitation, we propose the hybrid averaging operator based on Frank t-norm and t-conorm as follow:

Definition 3.19. Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j]) (j = 1, 2, \dots, n)$ be a collection of the IVPFFNs and let $\mathrm{IVPFFWA} : \Omega^n \to \Omega$, if

$$\begin{aligned}\mathrm{IVPFFHWA}_{\omega,\varpi}(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) &= \oplus_F{}_{j=1}^{n}\left(\omega_j \cdot_F \widehat{\widetilde{a}}_{\sigma(j)}\right) \\ &= \omega_1 \cdot_F \widehat{\widetilde{a}}_{\sigma(1)} \oplus_F \omega_2 \cdot_F \widehat{\widetilde{a}}_{\sigma(2)} \oplus_F \dots \oplus_F \omega_n \cdot_F \widehat{\widetilde{a}}_{\sigma(n)}\end{aligned} \tag{3.23}$$

then the function IVPFFHA is called Interval-Valued Picture Fuzzy Frank Hybrid Averaging (IVPFFHA) operator, where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of associated with IVPFFHA such that $\omega_j > 0$ and $\Sigma_{j=1}^n \omega_j = 1$, $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$ is weight vector of $\widetilde{a}_j (j = 1, 2, \dots, n)$ with $\varpi_j > 0$ and $\Sigma_{j=1}^n \varpi_j = 1$. Let $\widehat{\widetilde{a}}_j = (n\varpi_j) \cdot_F \widetilde{a}_j$; n is the adjustment factor. Suppose $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $\widehat{\widetilde{a}}_{\sigma(j-1)} \geq \widehat{\widetilde{a}}_{\sigma(j)}$ for all $j = 2, 3, \dots, n$.

Theorem 3.20. *Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j]) (j = 1, 2, \dots, n)$ be a collection of the IVPFFNs, then their aggregated value by IVPFFHA operator is still a IVPFFN, and*

$$\begin{aligned}\mathrm{IVPFFHA}_{\omega,\varpi}(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) &= \oplus_F{}_{j=1}^{n}\left(\omega_j \cdot_F \widehat{\widetilde{a}}_{\sigma(j)}\right) \\ &= \left(\left[1 - \log_\lambda\left(1 + \prod_{j=1}^n \left(\lambda^{1-\widehat{a}_{\sigma(j)}} - 1\right)^{w_j}\right), 1 - \log_\lambda\left(1 + \prod_{j=1}^n \left(\lambda^{1-\widehat{b}_{\sigma(j)}} - 1\right)^{w_j}\right)\right],\right. \\ &\quad \left[\log_\lambda\left(1 + \prod_{j=1}^n \left(\lambda^{\widehat{c}_{\sigma(j)}} - 1\right)^{w_j}\right), \log_\lambda\left(1 + \prod_{j=1}^n \left(\lambda^{\widehat{d}_{\sigma(j)}} - 1\right)^{w_j}\right)\right],\end{aligned}$$

$$\left[\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\widehat{e}_{\sigma(j)}}-1\right)^{w_j}\right),\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\widehat{f}_{\sigma(j)}}-1\right)^{w_j}\right)\right]\Bigg). \tag{3.24}$$

From Definition (3.19) we can know that the IVPFFHA operator firstly weights the given data, and then reorders the weighted values in descending order and weights these ordered data. Therefore, IVPFFHA operator can reflect the importance degrees of both the given data and their ordered positions.

Corollary 3.21. *The IVPFFWA operator and IVPFFOWA operator are special case of the IVPFFHA operator.*

Proof. Let $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then IVPFFHA operator will reduce to IVPFFWA operator and if $\varpi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then IVPFFHA operator will reduce to IVPFFOWA. □

4. Interval-valued picture fuzzy Frank geometric aggregation operators

Definition 4.1. Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs and let IVPFFWA : $\Omega^n \to \Omega$, if

$$\begin{aligned}\text{IVPFFWG}_\omega(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) &= \otimes_F{}_{j=1}^{n}\left(\widetilde{a}_j^{\wedge_F \omega_j}\right)\\ &= \widetilde{a}_1^{\wedge_F \omega_1} \otimes_F \widetilde{a}_2^{\wedge_F \omega_2} \ldots \otimes_F \widetilde{a}_n^{\wedge_F \omega_n}\end{aligned} \tag{4.1}$$

then the function IVPFFWG is called interval-valued picture fuzzy Frank Weighted Geometric (IVPFFWG) operator, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\widetilde{a}_j$ $(j = 1, 2, \ldots, n)$ such that $\omega_j > 0$ and $\Sigma_{j=1}^n \omega_j = 1$.

Theorem 4.2. *Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs, then their aggregated value by IVPFFWG operator is still a IVPFFN, and*

$$\begin{aligned}\text{IVPFFWG}_\omega(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) &= \otimes_F{}_{j=1}^{n}\left(\widetilde{a}_j^{\wedge_F \omega_j}\right)\\ &= \left(\left[\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{a_j}-1)^{w_j}\right),\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{b_j}-1)^{w_j}\right)\right],\right.\\ &\quad\left[1-\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{1-c_j}-1)^{w_j}\right),1-\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{1-d_j}-1)^{w_j}\right)\right],\\ &\quad\left.\left[1-\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{1-e_j}-1)^{w_j}\right),1-\log_\lambda\left(1+\prod_{j=1}^{n}(\lambda^{1-f_j}-1)^{w_j}\right)\right]\right)\end{aligned} \tag{4.2}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\widetilde{a}_j$ $(j = 1, 2, \ldots, n)$ such that $\omega_j > 0$ and $\Sigma_{j=1}^n \omega_j = 1$.

Theorem 4.3. *Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs and $\lambda > 1$. As $\lambda \to 1$, the IVPFFWG operator approaches the following limit*

$$\begin{aligned}&\lim_{\lambda\to 1}\text{IVPFFGWA}_\omega(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n)\\ &= \left(\left[\prod_{j=1}^{n}(a_j)^{w_j},\prod_{j=1}^{n}(b_j)^{w_j}\right],\left[1-\prod_{j=1}^{n}(1-c_j)^{w_j},1-\prod_{j=1}^{n}(1-d_j)^{w_j}\right],\left[1-\prod_{j=1}^{n}(1-e_j)^{w_j},1-\prod_{j=1}^{n}(1-f_j)^{w_j}\right]\right).\end{aligned}$$

Theorem 4.4. *Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs and $\lambda > 1$. As $\lambda \to \infty$, the IVPFFWG operator approaches the following limit*

$$\lim_{\lambda\to\infty} \mathrm{IVPFFWG}_{\omega}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) = \left(\left[\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j \right], \left[\sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j \right], \left[\sum_{j=1}^{n} w_j e_j, \sum_{j=1}^{n} w_j f_j \right] \right).$$

Theorem 4.5. *Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs and $\lambda > 1$, then the IVPFFWG operator is monotonically increasing with respect to λ.*

Definition 4.6. Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs and let $\mathrm{IVPFFWA} : \Omega^n \to \Omega$, if

$$\begin{aligned} \mathrm{IVPFFOWG}_{\omega}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) &= \otimes_{F}{}_{j=1}^{n} \left(\widetilde{a}_{\sigma(j)}^{\wedge_F \omega_j} \right) \\ &= \widetilde{a}_{\sigma(1)}^{\wedge_F \omega_1} \otimes_F \widetilde{a}_{\sigma(2)}^{\wedge_F \omega_2} \ldots \otimes_F \widetilde{a}_{\sigma(n)}^{\wedge_F \omega_n} \end{aligned} \tag{4.3}$$

then the function IVPFFOWG is called Interval-Valued Picture Fuzzy Frank Ordered Weighted Geometric (IVPFFOWG) operator, where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\widetilde{a}_{\sigma(j-1)} \geq \widetilde{a}_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\widetilde{a}_j (j = 1, 2, \ldots, n)$ such that $\omega_j > 0$ and $\Sigma_{j=1}^{n} \omega_j = 1$.

Theorem 4.7. *Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs, then their aggregated value by IVPFFOWG operator is still a IVPFFN, and*

$$\begin{aligned} \mathrm{IVPFFOWG}_{\omega}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) &= \otimes_{F}{}_{j=1}^{n} \left(\widetilde{a}_{\sigma(j)}^{\wedge_F \omega_j} \right) \\ &= \Bigg(\left[\log_{\lambda} \left(1 + \prod_{j=1}^{n} (\lambda^{a_{\sigma(j)}} - 1)^{w_j} \right), \log_{\lambda} \left(1 + \prod_{j=1}^{n} \left(\lambda^{b_{\sigma(j)}} - 1\right)^{w_j} \right) \right], \\ &\quad \left[1 - \log_{\lambda} \left(1 + \prod_{j=1}^{n} \left(\lambda^{1-c_{\sigma(j)}} - 1\right)^{w_j} \right), 1 - \log_{\lambda} \left(1 + \prod_{j=1}^{n} \left(\lambda^{1-d_{\sigma(j)}} - 1\right)^{w_j} \right) \right], \\ &\quad \left[1 - \log_{\lambda} \left(1 + \prod_{j=1}^{n} \left(\lambda^{1-e_{\sigma(j)}} - 1\right)^{w_j} \right), 1 - \log_{\lambda} \left(1 + \prod_{j=1}^{n} \left(\lambda^{1-f_{\sigma(j)}} - 1\right)^{w_j} \right) \right] \Bigg) \end{aligned} \tag{4.4}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\widetilde{a}_j$ $(j = 1, 2, \ldots, n)$ such that $\omega_j > 0$ and $\Sigma_{j=1}^{n} \omega_j = 1$.

Definition 4.8. Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs and let $\mathrm{IVPFFWA} : \Omega^n \to \Omega$, if

$$\begin{aligned} \mathrm{IVPFFHG}_{\omega,\varpi}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) &= \otimes_{F}{}_{j=1}^{n} \left(\widehat{\widetilde{a}}_{\sigma(j)}^{\wedge_F \omega_j} \right) \\ &= \widehat{\widetilde{a}}_{\sigma(1)}^{\wedge_F \omega_1} \otimes_F \widehat{\widetilde{a}}_{\sigma(2)}^{\wedge_F \omega_2} \otimes_F \ldots \otimes_F \widehat{\widetilde{a}}_{\sigma(n)}^{\wedge_F \omega_n} \end{aligned} \tag{4.5}$$

then the function IVPFFHG is called Interval-Valued Picture Fuzzy Frank Hybrid Geometric (IVPFFHG) operator, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of associated with IVPFFHG such that $\omega_j > 0$ and $\Sigma_{j=1}^{n} \omega_j = 1$, $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_n)^T$ is weight vector of $\widetilde{a}_j (j = 1, 2, \ldots, n)$ with $\varpi_j > 0$ and $\Sigma_{j=1}^{n} \varpi_j = 1$. Let $\widehat{\widetilde{a}}_j = \widetilde{a}_j^{\wedge_F n \omega_j}$; n is the adjustment factor. Suppose $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\widehat{\widetilde{a}}_{\sigma(j-1)} \geq \widehat{\widetilde{a}}_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$.

Theorem 4.9. *Let $\widetilde{a}_j = ([a_j, b_j], [c_j, d_j], [e_j, f_j])(j = 1, 2, \ldots, n)$ be a collection of the IVPFFNs, then their aggregated value by IVPFFHG operator is still a IVPFFN, and*

$$\begin{aligned}
&\mathrm{IVPFFHG}_{\omega,\varpi}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) = \otimes_F{}_{j=1}^{n}\left(\widehat{\widetilde{a}}_{\sigma(j)}^{\wedge_F \omega_j}\right)\\
&= \left(\left[\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\widehat{a}_{\sigma(j)}}-1\right)^{w_j}\right), \log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{\widehat{b}_{\sigma(j)}}-1\right)^{w_j}\right)\right],\right.\\
&\left[1-\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\widehat{c}_{\sigma(j)}}-1\right)^{w_j}\right), 1-\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\widehat{d}_{\sigma(j)}}-1\right)^{w_j}\right)\right],\\
&\left.\left[1-\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\widehat{e}_{\sigma(j)}}-1\right)^{w_j}\right), 1-\log_\lambda\left(1+\prod_{j=1}^{n}\left(\lambda^{1-\widehat{f}_{\sigma(j)}}-1\right)^{w_j}\right)\right]\right). \qquad (4.6)
\end{aligned}$$

Corollary 4.10. *The IVPFFWG operator and IVPFFOWG operator are special case of the IVPFFHG operator.*

Proof. Let $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then IVPFFHG operator will reduce to IVPFFWG operator and if $\varpi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then IVPFFHG operator will reduce to IVPFFOWG. □

5. Multiple attribute group decision-making methods on Frank aggregation operators

In this section, we will use these Frank aggregation operators for the MAGDM problems in which the attribute weights take the form of real numbers and attribute value take the form of IVFFNs.

(1) Description the decision-making problems.
For a MAGDM problem, let $E = \{E_1, E_2, \ldots, E_p\}$ be the collection of decision markers, $A = \{A_1, A_2, \ldots, A_m\}$ be the collection of alternatives, and $C = \{C_1, C_2, \ldots, C_n\}$ be the collection of attributes. Suppose that $\widetilde{a}_{ij}^k = ([a_{ij}^k, b_{ij}^k], [c_{ij}^k, d_{ij}^k], [e_{ij}^k, f_{ij}^k])$ is an attribute value given by the decision maker E_k, which is expressed by an IVPFN for the alternative A_i with respect to the attribute C_j, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weight vector of attribute set $C = C_1, C_2, \ldots, C_n$, and $\omega_j \in [0, 1], \sum_{j=1}^{n} \omega_j = 1$. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the weight vector of attribute set $E = E_1, E_2, \ldots, E_p$, and $\alpha_j \in [0, 1], \sum_{j=1}^{p} \alpha_j = 1$. Then, we use the attribute weight, the decision makers weights and the attribute values to rank the order of the alternatives.

(2) The first method based on the Frank hybrid averaging operator: the decision making steps of this method are shown as follows.

Step 1. Normalize the decision making information.

In general, for attribute values there are benefit attributes (I_1) (the bigger the attribute values the better) and cost attributes (I_2) (the smaller the attribute values the better). In order to eliminate the impact of different type attribute value, we need to normalize the decision making information. We may transform the attribute values from cost type to benefit type; in such a case, decision matrices $A^k = [\widetilde{a}_{ij}^k]_{m\times n}(k = 1, 2, \ldots p)$ can be transformed into matrices $R^k = [\widetilde{r}_{ij}^k]_{m\times n}(k = 1, 2, \ldots p)$, where

$$\widetilde{r}_{ij}^k = \left([r_{ij}^k, t_{ij}^k], [s_{ij}^k, q_{ij}^k], [g_{ij}^k, l_{ij}^k]\right) = \begin{cases} \left([a_{ij}^k, b_{ij}^k], [c_{ij}^k, d_{ij}^k], [e_{ij}^k, f_{ij}^k]\right), & C_j \in I_1 \\ \left([e_{ij}^k, f_{ij}^k], [c_{ij}^k, d_{ij}^k], [a_{ij}^k, b_{ij}^k]\right), & C_j \in I_2 \end{cases} \qquad (5.1)$$

for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 2. Utilize the IVPFFHA operator:

$$\widetilde{r}_{ij} = ([r_{ij}, t_{ij}], [s_{ij}, q_{ij}], [g_{ij}, l_{ij}]) = \mathrm{IVPFFHWA}\left(\widetilde{r}_{ij}^1, \widetilde{r}_{ij}^2, \ldots, \widetilde{r}_{ij}^p\right) \qquad (5.2)$$

to aggregate all the individual interval-valued picture fuzzy decision matrixes $R^k = [\widetilde{r}_{ij}^k]_{m\times n}(k = 1, 2, \dots p)$ in to the collective interval-valued picture fuzzy decision matrix $R = [\widetilde{r}_{ij}]_{m\times n}$.

Step 3. Utilize the IVPFFHA operator:

$$\widetilde{r}_i = ([r_i, t_i], [s_i, q_i], [g_i, l_i]) = \text{IVPFFHWA}(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \dots, \widetilde{r}_{in}) \tag{5.3}$$

to derive the collective overall preference values $\widetilde{r}_i(i = 1, 2, \dots, m)$.

Step 4. Calculate the score function $S(\widetilde{r}_i)(i = 1, 2, \dots, m)$ of the collective overall values $\widetilde{r}_i(i = 1, 2, \dots, m)$, and then, rank all the alternatives $A_1, A_2, \dots, A_m$. When two score functions $S(\widetilde{r}_i)$ and $S(\widetilde{r}_j)$ are equal, we need to calculate their accuracy functions $H(\widetilde{r}_i)$ and $H(\widetilde{r}_j)$, and then, we can rank them by accuracy functions.

Step 5. Rank the alternatives.
Rank the alternatives $A_1, A_2, \dots, A_m$ and select the best one(s) by score function $S(\widetilde{r}_i)$ and accuracy function $H(\widetilde{r}_i)$.

Step 6. End.

(3) The second method based on the Frank geometric hybrid weighted averaging operator: the decision making steps of this method are shown as follows.

Step 1. Normalize the decision making information.
It is the same as step 1 in first method.

Step 2. Utilize the IVPFFHG operator:

$$\widetilde{r}_{ij} = ([r_{ij}, t_{ij}], [s_{ij}, q_{ij}], [g_{ij}, l_{ij}]) = \text{IVPFFGHWA}\left(\widetilde{r}_{ij}^1, \widetilde{r}_{ij}^2, \dots, \widetilde{r}_{ij}^p\right) \tag{5.4}$$

to aggregate all the individual interval-valued picture fuzzy decision matrixes $R^k = [\widetilde{r}_{ij}^k]_{m\times n}(k = 1, 2, \dots p)$ in to the collective interval-valued picture fuzzy decision matrix $R = [\widetilde{r}_{ij}]_{m\times n}$.

Step 3. Utilize the IVPFFHG operator:

$$\widetilde{r}_i = ([r_i, t_i], [s_i, q_i], [g_i, l_i]) = \text{IVPFFGHWA}(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \dots, \widetilde{r}_{in}) \tag{5.5}$$

to derive the collective overall preference values $\widetilde{r}_i(i = 1, 2, \dots, m)$.

Step 4. It is the same as step 4 in first method.

Step 5. It is the same as step 5 in first method.

Step 6. End.

6. Application example

In order to demonstrate the application of the proposed method, we will cite an example about the air quality evaluation (adapted from [34]). To evaluate the air quality of Tehran for the football games held during September 10-30, 2019, the air quality in Tehran for the September of 2015, 2016, 2017 and 2018 were collected in order to find out the trends and to forecast the situation in September 2019. There are 3 air-quality monitoring stations (E_1, E_2, E_3) which can be seen as decision-makers, and their weight is $(0.30, 0.40, 0.30)^T$. There are measured indices, namely $SO_2(C_1)$, $NO_2(C_2)$, $PM_{10}(C_3)$, and their weight is $(0.40, 0.25, 0.35)^T$. The measured values from air-quality monitoring stations under these indices are shown in Tables 1–3, and they can be expressed by interval-valued picture fuzzy numbers.

Let $(A_1, A_2, A_3, A_4) = \{$September of 2015, September of 2016, September of 2017, September of 2018$\}$ be the set of alternatives. The rank of air quality from 2015 to 2018 can be found as follows.

To obtain the best alternative(s), the following steps are involved:

Step 1. Normalize the decision-making information. Since all the measured values are of the same type, they do not need normalization.

Table 1. Air quality data from station E_1.

	$C_1(\mathrm{SO}_2)$	$C_2(\mathrm{NO}_2)$	$C_3(\mathrm{PM}_{10})$
$A_1(2015)$	$([0.12, 0.40], [0.24, 0.33], [0.15, 0.24])$	$([0.15, 0.25], [0.21, 0.35], [0.18, 0.23])$	$([0.10, 0.25], [0.12, 0.32], [0.08, 0.24])$
$A_2(2016)$	$([0.20, 0.25], [0.05, 0.25], [0.15, 0.35])$	$([0.10, 0.22], [0.13, 0.33], [0.20, 0.30])$	$([0.17, 0.26], [0.25, 0.30], [0.33, 0.38])$
$A_3(2017)$	$([0.35, 0.41], [0.20, 0.28], [0.21, 0.25])$	$([0.05, 0.20], [0.25, 0.30], [0.08, 0.25])$	$([0.15, 0.23], [0.28, 0.33], [0.20, 0.28])$
$A_4(2018)$	$([0.08, 0.22], [0.12, 0.26], [0.22, 0.32])$	$([0.18, 0.28], [0.13, 0.20], [0.05, 0.32])$	$([0.10, 0.27], [0.18, 0.25], [0.22, 0.32])$

Table 2. Air quality data from station E_2.

	$C_1(\mathrm{SO}_2)$	$C_2(\mathrm{NO}_2)$	$C_3(\mathrm{PM}_{10})$
$A_1(2015)$	$([0.15, 0.32], [0.22, 0.30], [0.25, 0.34])$	$([0.23, 0.28], [0.24, 0.33], [0.10, 0.26])$	$([0.05, 0.23], [0.25, 0.32], [0.21, 0.28])$
$A_2(2016)$	$([0.22, 0.31], [0.28, 0.32], [0.21, 0.25])$	$([0.05, 0.23], [0.20, 0.31], [0.18, 0.25])$	$([0.18, 0.24], [0.18, 0.30], [0.22, 0.25])$
$A_3(2017)$	$([0.28, 0.33], [0.08, 0.21], [0.30, 0.35])$	$([0.18, 0.28], [0.23, 0.30], [0.05, 0.32])$	$([0.21, 0.27], [0.18, 0.25], [0.17, 0.30])$
$A_4(2018)$	$([0.09, 0.20], [0.16, 0.25], [0.22, 0.35])$	$([0.20, 0.26], [0.23, 0.30], [0.10, 0.25])$	$([0.33, 0.36], [0.21, 0.27], [0.13, 0.18])$

Table 3. Air quality data from station E_3.

	$C_1(\mathrm{SO}_2)$	$C_2(\mathrm{NO}_2)$	$C_3(\mathrm{PM}_{10})$
$A_1(2015)$	$([0.08, 0.30], [0.14, 0.23], [0.25, 0.29])$	$([0.25, 0.34], [0.23, 0.31], [0.08, 0.21])$	$([0.18, 0.27], [0.19, 0.35], [0.20, 0.26])$
$A_2(2016)$	$([0.13, 0.28], [0.15, 0.25], [0.20, 0.30])$	$([0.11, 0.22], [0.13, 0.21], [0.09, 0.20])$	$([0.07, 0.26], [0.15, 0.20], [0.33, 0.40])$
$A_3(2017)$	$([0.25, 0.38], [0.22, 0.29], [0.07, 0.25])$	$([0.25, 0.30], [0.20, 0.29], [0.18, 0.35])$	$([0.25, 0.32], [0.18, 0.23], [0.19, 0.37])$
$A_4(2018)$	$([0.28, 0.20], [0.17, 0.23], [0.21, 0.32])$	$([0.09, 0.18], [0.21, 0.27], [0.25, 0.31])$	$([0.17, 0.23], [0.28, 0.35], [0.26, 0.31])$

Step 2. Utilize the IVIPFFHA operator expressed by equation (3.24) to aggregate all the individual interval-valued picture fuzzy decision matrices into the collective decision matrix. We can obtain (suppose $\lambda = 2$, $\omega = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$).

Therefore, we can get

$$R = \begin{pmatrix} \widetilde{r}_{11} & \widetilde{r}_{12} & \widetilde{r}_{13} \\ \widetilde{r}_{21} & \widetilde{r}_{22} & \widetilde{r}_{23} \\ \widetilde{r}_{31} & \widetilde{r}_{32} & \widetilde{r}_{33} \\ \widetilde{r}_{41} & \widetilde{r}_{42} & \widetilde{r}_{43} \end{pmatrix}$$

such that

$$\begin{aligned}
\widetilde{r}_{11} &= ([0.1203, 0.3391], [0.1975, 0.2853], [0.2148, 0.2922]) \\
\widetilde{r}_{12} &= ([0.2125, 0.2905], [0.2540, 0.3295], [0.1118, 0.2351]) \\
\widetilde{r}_{13} &= ([0.1359, 0.2482], [0.1852, 0.3287], [0.1556, 0.2614]) \\
\widetilde{r}_{21} &= ([0.1876, 0.2833], [0.1335, 0.2761], [0.1872, 0.2923]) \\
\widetilde{r}_{22} &= ([0.0835, 0.2240], [0.1546, 0.2815], [0.1513, 0.2472]) \\
\widetilde{r}_{23} &= ([0.1450, 0.2528], [0.1881, 0.2660], [0.2811, 0.3272]) \\
\widetilde{r}_{31} &= ([0.2928, 0.3697], [0.1434, 0.2524], [0.1759, 0.2864]) \\
\widetilde{r}_{32} &= ([0.1646, 0.2628], [0.2267, 0.2968], [0.0850, 0.3055]) \\
\widetilde{r}_{33} &= ([0.2047, 0.2737], [0.2058, 0.2652], [0.1843, 0.3131]) \\
\widetilde{r}_{41} &= ([0.1476, 0.2065], [0.1495, 0.2467], [0.2169, 0.3317])
\end{aligned}$$

TABLE 4. Ordering the alternatives by utilizing different values of λ.

λ	Score function $S(\widetilde{r}_i)$		Ranking
$\lambda \to 1$	$S(\widetilde{r}_1) = -0.2497$	$S(\widetilde{r}_2) = -0.2589$	$A_3 \succ A_4 \succ A_1 \succ A_2$
	$S(\widetilde{r}_3) = -0.1763$	$S(\widetilde{r}_4) = -0.2385$	
$\lambda = 2$	$S(\widetilde{r}_1) = -0.2502$	$S(\widetilde{r}_2) = -0.2607$	$A_3 \succ A_4 \succ A_1 \succ A_2$
	$S(\widetilde{r}_3) = -0.1789$	$S(\widetilde{r}_4) = -0.2423$	
$\lambda = 5$	$S(\widetilde{r}_1) = -0.2520$	$S(\widetilde{r}_2) = -0.2630$	$A_3 \succ A_4 \succ A_1 \succ A_2$
	$S(\widetilde{r}_3) = -0.1810$	$S(\widetilde{r}_4) = -0.2441$	
$\lambda = 50$	$S(\widetilde{r}_1) = -0.2554$	$S(\widetilde{r}_2) = -0.2659$	$A_3 \succ A_4 \succ A_1 \succ A_2$
	$S(\widetilde{r}_3) = -0.1858$	$S(\widetilde{r}_4) = -0.2492$	
$\lambda \to \infty$	$S(\widetilde{r}_1) = -0.2766$	$S(\widetilde{r}_2) = -0.2841$	$A_3 \succ A_4 \succ A_1 \succ A_2$
	$S(\widetilde{r}_3) = -0.2007$	$S(\widetilde{r}_4) = -0.2621$	

$$\widetilde{r}_{42} = ([0.1618, 0.2427], [0.1889, 0.2576], [0.1077, 0.2901])$$
$$\widetilde{r}_{43} = ([0.2178, 0.2957], [0.2188, 0.2854], [0.1878, 0.2520]).$$

Step 3. Utilize the IVIPFFHA operator expressed by equation (3.24) to derive the collective overall preference values (suppose $\lambda = 2$, $\omega = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$). Therefore, we can get

$$\widetilde{r}_1 = ([0.1494, 0.2959], [0.2057, 0.3108], [0.1632, 0.2662])$$
$$\widetilde{r}_2 = ([0.1473, 0.2581], [0.1562, 0.2738], [0.2051, 0.2917])$$
$$\widetilde{r}_3 = ([0.2313, 0.3107], [0.1827, 0.2674], [0.1494, 0.3003])$$
$$\widetilde{r}_4 = ([0.1761, 0.2475], [0.1812, 0.2624], [0.1732, 0.2915]).$$

Step 4. Calculate the score function $S(\widetilde{r}_i)(i = 1, 2, 3, 4)$ of the collective overall values $\widetilde{r}_i(i = 1, 2, 3, 4)$, then we can obtain:

$$S(\widetilde{r}_1) = -0.2502, \qquad S(\widetilde{r}_2) = -0.2607, \qquad S(\widetilde{r}_3) = -0.1789, \qquad S(\widetilde{r}_4) = -0.2423.$$

Step 5. According to the score function $S(\widetilde{r}_i)(i = 1, 2, 3, 4)$, rank the alternatives $\{A_1, A_2, A_3, A_4\}$ shown as follows:

$$A_3 \succ A_4 \succ A_1 \succ A_2.$$

Therefore, the best alternative is A_3, *i.e.*, the beast air quality in Tehran is September of 2017 among the September 2015, 2016, 2017 and 2018.

Influence of the parameter on decision making result of this example. In order to illustrate the influence of the parameter λ on decision making of this example, we use the different values of λ in steps 2 and 3 to rank the alternatives. The ranking results are shown in Table 4. As we see from Table 4, the aggregation results using the different aggregation parameter λ are different, but, the orderings of the alternatives are same in this example. In general, we can regard parameter λ as the attitude of decision makers; the more the value of parameter λ is, the more optimistic attitude is. In real decision making, we can select the specific parameter λ according to real decision making problem.

Acknowledgements. We would like to thank the editor and reviewers for their thoughtful comments and efforts towards improving our paper.

REFERENCES

[1] K.T. Atanassov, Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **20** (1986) 87–96.

[2] K.T. Atanassov, More on intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **33** (1989) 37–46.
[3] K.T. Atanassov, Operators over interval-valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **64** (1994) 195–174.
[4] K.T. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **31** (1989) 343–349.
[5] G. Beliakov, A. Pradera and T. Calvo, Aggregation Functions: A Guide for practiyioners. Springer, Heidelberg, Germany (2007).
[6] T. Calov, B. De Beats and J. Fodor, The functional equations of Frank and Alsina for uninorms and nullnorms. *Fuzzy Sets Syst.* **120** (2001) 385–394.
[7] J. Casanovas and J. Torrens, An axiomatic approach to fuzzy cardinalities of finite fuzzy sets. *Fuzzy Sets Syst.* **133** (2003) 193–209.
[8] C.T. Chen, Extensions of the TOPSIS for group cecision-making under fuzzy environment. *Fuzzy Sets Syst.* **114** (2000) 1–9.
[9] S.M. Chen, L.W. Lee, H.C. Liu and S.W. Yang, Multiattribute decision making based on interval-valued intuitionistic fuzzy values. *Expert Syst. App.* **39** (2012) 10343–10351.
[10] B.C. Cuong, Picture fuzzy sets-first results, part 1. Seminar neuro-fuzzy systems with applications. Tech. rep., Instiute of mathematics, Hanoi (2013).
[11] B.C. Cuong, Picture fuzzy sets-first results, part 2. Seminar neuro-fuzzy systems with applications. Tech. rep., Instiute of mathematics, Hanoi (2013).
[12] G. Deschrijver, Generalized arithmetic operators and their relationship to t-norms in interval-valued fuzzy set theory. *Fuzzy Sets Syst.* **160** (2009) 3080–3102.
[13] G. Deschrijver and E.E. Kerre, Uninorms in L*-fuzzy set theory. *Fuzzy Sets Syst.* **148** (2004) 243–262.
[14] M.J. Frank, On the simultaneous associativity of $F(x,y)$ and $x+y-F(x,y)$. *Aequationes Mathematicae* **19** (1979) 194–226.
[15] H. Garg, Generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision making. *Comput. Ind. Eng.* **101** (2016) 53–69.
[16] H. Garg, N. Agarwal and A. Tripathi, Generalized intuitionistic fuzzy entropy measure of order α and β degree and its appliction to multi-criteria decision making problem. *Int. J. Fuzzy Syst. App.* **6** (2017) 86–107.
[17] J. Hu, Y. Zhang, X. Chen and Y. Liu, Multi-criteria decision making method based on possibility degree of interval type-2 fuzzy number. *Knowl. Based Syst.* **43** (2013) 21–29.
[18] K. Kumar and H. Garg, TOPSIS method based on the connection number of set pair analysis under interval-valued intuitionistic fuzzy set environment. *Comput. Appl. Math.* **37** (2016) 1319–1329.
[19] P.D. Liu, Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. *IEEE Trans. Fuzzy Syst.* **22** (2014) 83–97.
[20] P.D. Liu, Y.H. Li and Y.B. Chen, Some generalized Einstein aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. *Scintia Iranica* **22** (2015) 2684–2701.
[21] J. Qin, X. Liu and W. Pedrycz, Frank aggregation operators and their application to hesitant fuzzy multiple attribute decision making. *Appl. Soft Comput.* **41** (2016) 428–452.
[22] P. Sarkoci, Domination in the families of Frank and Hamacher t-norms. *Kybernetika* **41** (2005) 349–360.
[23] L.H. Son, Generalized picture distance measure and applications to picture fuzzy clustering. *Appl. Soft Comput.* **46** (2016) 284–295.
[24] W. Wang and H. He, Research on flexible probability logic operator based on Frank T/S norms. *Acta Electron. Sinic.* **37** (2009) 1141–1145.
[25] W. Wang and X. Liu, Intuitionistic fuzzy aggregation using Einstein operations. *IEEE Trans. Fuzzy Syst.* **20** (2012) 923–938.
[26] J.Q. Wang, K.J. Li and H.Y. Zhang, Interval-valued intuitionistic fuzzy multi-criteria decision-making aproach based on prospect score function. *Knowl.-Based Syst.* **20** (2012) 119–125.
[27] J.Q. Wang, R.R. Nie, H.Y. Zhang and X.H. Chen, Intuitionistic fuzzy multi-criteria decision-making method based on evidential reasoning. *Appl. Soft Comput.* **13** (2013) 1823–1831.
[28] G.W. Wei, picture fuzzy cross-entropy for multiple attribute decision making problems. *J. Bus. Econ. Manage.* **17** (2016) 491–502.
[29] M. Xia, Z. Xu and B. Zhu, Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm. *Knowl. Based Syst.* **31** (2012) 78–88.
[30] Z. Xu, An overview of methods for determining weights. *Int. J. Intell. Syst.* **20** (2005) 843–865.
[31] Z. Xu, Multi-person malti-attribute decision making models under intuitionistics fuzzy environment. *Fuzzy Optim. Decis. Mak.* **7** (2007) 221–236.
[32] Z. Xu and R.R. Yager, Intuitionistic and interval-valued intuitionistic fuzzy preference relations and their measures of similarity for the evaluation of agreement within a group. *Fuzzy Optim. Decis. Mak.* **8** (2009) 123–139.
[33] R.R. Yager, On some new classes of implication operators and their role in approximate reasoning. *Inf. Sci.* **167** (2004) 193–216.
[34] Z.L. Yue, Deriving decision maker's weights based on distance measur for interval-valued intuitionistic fuzzy group decision making. *Expert Syst. App.* **38** (2011) 11665–11670.
[35] L.A. Zadeh, Fuzzy sets. *Inf. Control* **8** (1965) 338–353.