ON THE SUPER CONNECTIVITY OF DIRECT PRODUCT OF GRAPHS

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Abstract. A vertex-cut S is called a *super vertex-cut* if $G-S$ is disconnected and it contains no isolated vertices. The *super-connectivity*, $\kappa^{'}$, is the minimum cardinality over all super vertex-cuts. This article provides bounds for the super connectivity of the direct product of an arbitrary graph and the complete graph K_n. Among other results, we show that if G is a non-complete graph with girth$(G)=3$ and $\kappa^{'}(G)=\infty$, then $\kappa^{'}(G\times K_n)\leq \min\{mn-6, m(n-1)+5, 5n+m-8\}$, where $|V(G)|=m$.

1. Introduction

We follow [1] for graph theoretic terminologies and notations not defined here. Let G be a simple undirected graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. For two vertices $u,v\in V(G)$, u and v are *neighbors* if u and v are adjacent and we write $u\sim v$. If u and v are not adjacent in G, then we write $uv\notin E(G)$. For each vertex $v\in V(G)$, the *neighborhood* $N_G(v)$ of v is defined as the set of all vertices adjacent to v and $deg(v)=|N_G(v)|$ is the *degree* of v. The number $\delta(G)=\min\{\deg(v)\mid v\in V(G)\}$ is the minimum degree of G. Also the *girth* of the graph G, $girth(G)$, is the length of its shortest cycle if G contains cycle, define girth$(G)=\infty$ otherwise. A *wheel* graph is a graph formed by connecting a single universal vertex to all vertices of a cycle. We use W_n to denote a wheel graph with $n\geq 3$ vertices. For an arbitrary subset $S\subset V(G)$ we use $G-S$ to denote the graph obtained by removing all vertices in S from G. For any connected graph G, if $G-S$ is disconnected, then S is a *vertex-cut*. The *connectivity* of a graph G, denoted by $\kappa(G)$, is the minimum cardinality of a set $S\subset V(G)$ such that $G-S$ is either disconnected or the trivial graph K_1. It is known that $\kappa(G)\leq\delta(G)$. A vertex-cut S is called a *super vertex-cut* if $G-S$ is disconnected and it contains no isolated vertices. The *super-connectivity* $\kappa^{'}$ is the minimum cardinality over all super vertex-cuts, that is,

$$\kappa^{'}(G)=\min\{|S|\mid S\subseteq V \text{ is a super vertex-cut of G}\}$$

Clearly, the super connectivity $\kappa^{'}(G)$ does not always exist for a connected graph G. We write $\kappa^{'}(G)=\infty$ if $\kappa^{'}(G)$ does not exist. For example, $\kappa^{'}(G)=\infty$ if G is the star $K_{1,n}$.

It is well known that when the underlying topology of an interconnection network is modeled by a graph $G=(V,E)$, where V represents the set of processors and E

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represents the set of communication links in the network, the connectivity $\kappa(G)$ of G is an important measurement for the fault tolerance of the network. It has been shown that a super connected network is most reliable and has the smallest vertex failure rate among all the networks with the same connectivity (see, for example [20, 21]).

The direct product $X \times Y$ of two graphs X and Y is the graph having $V(X \times Y) = V(X) \times V(Y)$ and $E(X \times Y) = \{(x_1, y_1)(x_2, y_2) \mid x_1x_2 \in E(X) \text{ and } y_1y_2 \in E(Y)\}$.

We state two known results of the direct product of graphs that will be used in the proof of our main results.

Proposition 1.1. *([17]) Let G and H be connected graphs. The graph $G \times H$ is connected if and only if G or H contains an odd cycle.*

Proposition 1.2. *([17]) Let G be a connected graph. If G has no odd cycle, then $G \times K_2$ has exactly two components isomorphic to G.*

The direct product plays an important role in design and analysis of network [18]. This product has generated a lot of interest mainly due to its various applications. For instance, it is used in complex networks to generate realistic networks [12], in multiprocessor systems to model of concurrency [11] and in automata theory [5]. The connectivity of direct product graphs has been investigated in [15] and [17]. Also the connectivity of direct product of a bipartite graph and a complete graph has been presented by Guji and Vumar(see [6]). Moreover, the super connectivity of $K_{m,r} \times K_n$ is determined by Ekinci and Kirlangiç (see [3]). For more results we refer the reader to [2, 4, 7, 8, 9, 10, 13, 14, 16, 19, 22]. In this paper we investigate the super connectivity $\kappa^{'}$ of the direct product of an arbitrary graph and the complete graph K_n. We show that if $\kappa^{'}(G) = t < \infty$ then $\kappa^{'}(G \times K_n) \leq tn$. Also if $\kappa^{'}(G) = \infty$ and girth$(G) = 3$, then $\kappa^{'}(G \times K_n) \leq \min\{mn - 6, m(n-1) + 5, 5n + m - 8\}$, where $|V(G)| = m$.

2. SUPER CONNECTIVITY OF $G \times K_n$

Throughout this section, G is a connected non-complete graph.

Let G be a graph with $V(G) = \{x_1, x_2, \cdots, x_m\}$ and $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Suppose that $S_i = V(G) \times v_i$ for $i \in \mathbb{Z}_n$, where $\mathbb{Z}_n = \{1, 2, 3, \cdots, n\}$. Hence $V(G \times K_n) = S_1 \cup S_2 \cup \cdots \cup S_n$ where $\{S_i\}$ is a partition of $G \times K_n$.

Theorem 2.1. *Let G be a graph with $\kappa^{'}(G) = t < \infty$. Then $\kappa^{'}(G \times K_n) \leq tn$.*

Proof. Let $X = \{x_1, x_2, \cdots, x_t\}$ be a minimum super vertex-cut of G. Then $S = \{(x_j, v_i) \mid j \in \mathbb{Z}_t, i \in \mathbb{Z}_n\}$ is a super vertex-cut in $G \times K_n$. So $\kappa^{'}(G \times K_n) \leq tn$. ■

Let C_n be a cycle of length $n \geq 6$ and $V(C_n) = \{x_1, \cdots, x_n\}$ where $x_1 \sim x_2 \sim x_3 \sim \cdots \sim x_n \sim x_1$. Then $S = \{x_1, x_4\}$ is a super vertex-cut in C_n. Hence, for a graph G with girth$(G) \geq 6$ we have $\kappa^{'}(G) < \infty$ and by Theorem 2.1, $\kappa^{'}(G \times K_n) < \infty$. Thus we may suppose that girth$(G) \leq 5$. First suppose that girth$(G) = 5$. If $|E(G)| \geq 6$ then $\kappa^{'}(G) < \infty$ and so again by Theorem 2.1, $\kappa^{'}(G \times K_n) < \infty$. Thus in the following theorem we may suppose that girth$(G) = 5$ and $|E(G)| = 5$. It is easy to see that $\kappa^{'}(G \times K_2) = 2$.

Theorem 2.2. *Let G be a cycle of length* 5*. Then* $\kappa'(G \times K_n) = \min\{5n-8, 3n\}$ *for* $n \geq 3$.

Proof. Suppose that G is a cycle of length 5 and $V(G) = \{a, b, c, d, e\}$ where $a \sim b \sim c \sim d \sim e \sim a$. Let $j \in \mathbb{Z}_n$ be constant. Then by deleting all vertices $\{(a, v_j), (b, v_i), (c, v_j), (d, v_t), (e, v_t) \mid i \in \mathbb{Z}_n, t \in \mathbb{Z}_n - \{j\}\}$ we obtain a disconnected graph without any isolated vertex. Therefore, $\kappa'(G \times K_n) \leq 1 + n + 1 + 2(n-1) = 3n$. Now, let S be a super vertex-cut of $G \times K_n$. Hence $(G \times K_n) - S$ has at least two components, say C_1, C_2. Let $(x, v_r) \in C_1$ and $(y, v_t) \in C_2$ for some $x, y \in V(G)$. We have four cases:

Case 1. Let $x = y$. Hence $v_r \neq v_t$. Without loss of generality, let $x = a$. So $(a, v_r) \in C_1$ and $(a, v_t) \in C_2$. Since $N_G(a) = \{b, e\}$, for every $j \in \mathbb{Z}_n - \{r, t\}$, $(a, v_r) \sim (b, v_j) \sim (a, v_t)$ and $(a, v_r) \sim (e, v_j) \sim (a, v_t)$ are paths between (a, v_r) and (a, v_t) in $G \times K_n$. Therefore $\{(b, v_j), (e, v_j) \mid j \in \mathbb{Z}_n - \{r, t\}\} \subset S$. Clearly $(b, v_t), (e, v_t) \in C_1$ and $(b, v_r), (e, v_r) \in C_2$. Also, for every $j \in \mathbb{Z}_n - \{r, t\}$, $(b, v_t) \sim (a, v_j) \sim (b, v_r)$ and $(e, v_r) \sim (a, v_j) \sim (e, v_t)$ are paths in $G \times K_n$. Thus, $\{(a, v_j) \mid j \in \mathbb{Z}_n - \{r, t\}\} \subset S$. Similarly, $\{(d, v_j), (c, v_j) \mid j \in \mathbb{Z}_n - \{r, t\}\} \subset S$. By deleting these vertices of $G \times K_n$ we obtain an 10−gone, say P_{10}, where

$$V(P_{10}) = \{(a, v_t), (b, v_r), (c, v_t), (d, v_r), (e, v_t), (a, v_r), (b, v_t), (c, v_r), (d, v_t), (e, v_r)\}.$$

Hence $\bigcup_{i=1(i \neq r,t)}^{n} S_i$ with two more vertices is S, that is, $|S| = 5(n-2) + 2 = 5n - 8$.

Case 2. Let $x \neq y$, $v_r = v_t$ and $x \sim y$. Without loss of generality, let $x = a$ and $y = b$. Hence $S = \{(a, v_j), (b, v_j) \mid j \in \mathbb{Z}_n - \{r\}\} \cup \{(e, v_r), (c, v_r)\} \cup \{(d, v_i) \mid i \in \mathbb{Z}_n\}$. Therefore, in this case $|S| = 2(n-1) + 2 + n = 3n$.

Case 3. Let $x \neq y$, $v_r = v_t$ and $xy \notin E(G)$. Without loss of generality, let $x = a$ and $y = c$. Since $(a, v_r) \sim (b, v_j) \sim (c, v_r)$ is a path in $G \times K_n$, for every $v_j \neq v_r$, the set $\{(b, v_j) \mid j \in \mathbb{Z}_n - \{r\}\}$ lies in S. Now, we choose one element $l \in \mathbb{Z}_n$ with $l \neq r$. Since $(e, v_l) \in C_1$, $(d, v_l) \in C_2$ and $(e, v_i) \sim (d, v_j)$ for $i \neq j$, the set $\{(e, v_j), (d, v_j) \mid j \in \mathbb{Z}_n - \{l\}\}$ lies in S. In the remaining graph the vertex (b, v_r) is adjacent to all vertices of $\{(a, v_j) \in C_1, (c, v_j) \in C_2 \mid j \neq r\}$, so $(b, v_r) \in S$. Finally, (a, v_l) and (c, v_l) are isolated vertices. Thus they belong to S. Therefore, in this case $|S| = n + 2 + 2(n-1) = 3n$.

Case 4. Let $x \neq y$, $v_r \neq v_t$. Clearly $xy \notin E(G)$. Suppose that $x = a$, $y = c$ and $l \neq r, t$. Thus $(e, v_l) \in C_1$ and $(d, v_l) \in C_2$. Now with the similar arguments in Case 3, we get $S = \{(e, v_j), (d, v_j) \mid j \in \mathbb{Z}_n - \{l\}\} \cup \{(b, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(a, v_l), (c, v_l)\}$. Therefore, in this case $|S| = 3n$.

Thus $\kappa'(G \times K_n) \leq \min\{5n-8, 3n\}$. Furthermore, by the process of the proof, in all cases if $|S| < \min\{5n-8, 3n\}$ then $(G \times K_n) - S$ is either connected or has some isolated vertices. Therefore, $\kappa'(G \times K_n) = \min\{5n-8, 3n\}$. ∎

By the above theorem, the following result holds.

Corollary 2.3. *Let G be a cycle of length* 5*. Then* $\kappa'(G \times K_n) = 3n$ *for* $n \geq 4$.

Suppose that G is a bipartite graph with $\kappa'(G) = \infty$. Hence girth$(G) = \infty$ or girth(G) is even. If girth$(G) \geq 6$ then $\kappa'(G) < \infty$ and by Theorem 2.1, $\kappa'(G \times K_n) < \infty$. So we have the following result when girth$(G) = 4$ or girth$(G) = \infty$.

Theorem 2.4. *Let G be a bipartite graph and $\kappa^{'}(G) = \infty$. Then $\kappa^{'}(G \times K_n) \leq m(n-2)$, where $|V(G)| = m$.*

Proof. By Proposition 1.2, $(G \times K_n) - (\cup_{i=3}^{n} S_i) \cong G \times K_2$ has two components isomorphic to G. Thus $\kappa^{'}(G \times K_n) \leq m(n-2)$. ■

Finally in Theorem 2.5, we investigate $\kappa^{'}(G \times K_n)$ when girth$(G) = 3$ and $\kappa^{'}(G) = \infty$.

Figure.1

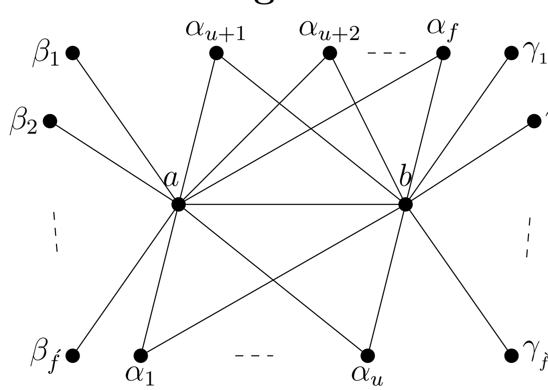
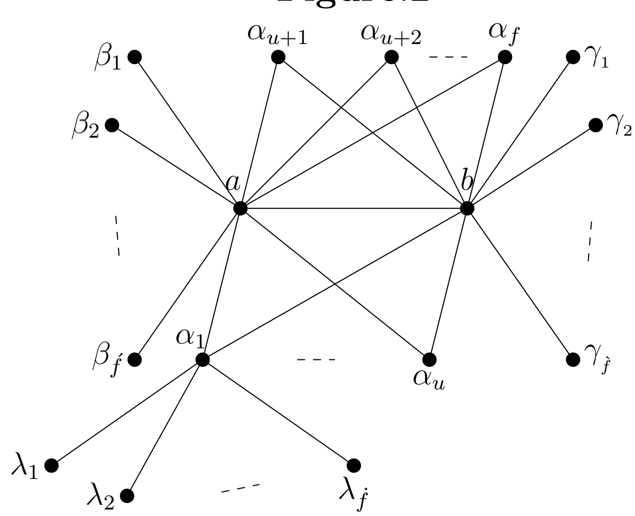


Figure.2



Theorem 2.5. *Let G be a graph with girth$(G) = 3$, $|V(G)| = m$ and $\kappa^{'}(G) = \infty$. Then $\kappa^{'}(G \times K_n) \leq min\{mn-6, m(n-1)+5, 5n+m-8\}$.*

Proof. First suppose that G has a unique triangle. Let C be the only cycle of G with $V(C) = \{u_1, u_2, u_3\}$. We consider the following cases:
Case 1. Let $\deg(u_j) = 2$ for all $j \in \mathbb{Z}_3$. Then G is K_3 and by [3, Theorem 2.7] $\kappa^{'}(G \times K_n) = 3n - 4$.

Case 2. Let $\deg(u_j) \geq 3$ for all $j \in \mathbb{Z}_3$. Also let $N_{G-\{u_2,u_3\}}(u_1) = \{x_{r'} \mid r' \in \mathbb{Z}_r\}$, $N_{G-\{u_1,u_3\}}(u_2) = \{y_{s'} \mid s' \in \mathbb{Z}_s\}$ and $N_{G-\{u_1,u_2\}}(u_3) = \{w_{t'} \mid t' \in \mathbb{Z}_t\}$ where $r+s+t = m-3$. Now, $\kappa'(G) = \infty$ implies that $\deg(x_{r'}) = \deg(y_{s'}) = \deg(w_{t'}) = 1$, for every $r' \in \mathbb{Z}_r$, $s' \in \mathbb{Z}_s$ and $t' \in \mathbb{Z}_t$. Hence $S = \{(u_j, v_l) \mid j \in \mathbb{Z}_3, l \in \mathbb{Z}_n - \{1\}\} \cup \{(x_{r'}, v_1), (y_{s'}, v_1), (w_{t'}, v_1) \mid r' \in \mathbb{Z}_r, s' \in \mathbb{Z}_s, t' \in \mathbb{Z}_t\}$ is a super vertex-cut in $G \times K_n$ with $|S| = 3(n-1) + (m-3) = 3n + m - 6$.

Case 3. Let $\deg(u_1) \geq 3$ and $\deg(u_2) = \deg(u_3) = 2$. In this case $S = \cup_{i=3}^{n} S_i \cup \{(u_3, v_1), (u_3, v_2)\}$ is a super vertex-cut in $G \times K_n$ with $|S| = m(n-2) + 2$.

Case 4. Let $\deg(u_1) \geq 3$, $\deg(u_2) \geq 3$ and $\deg(u_3) = 2$. Let $N_{G-\{u_2,u_3\}}(u_1) = \{x_{r'} \mid r' \in \mathbb{Z}_r\}$ and $N_{G-\{u_1,u_3\}}(u_2) = \{y_{s'} \mid s' \in \mathbb{Z}_s\}$ where $r + s = m - 3$. We have two subcases:

Subcase 1. If $\deg(x_{r'}) = \deg(y_{s'}) = 1$ for every $x_{r'} \in N_{G-\{u_2,u_3\}}(u_1)$ and $y_{s'} \in N_{G-\{u_1,u_3\}}(u_2)$ then $S = \{(u_3, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(u_1, v_j), (u_2, v_j) \mid j \in \mathbb{Z}_n - \{1\}\} \cup \{(y_{s'}, v_1), (x_{r'}, v_1) \mid r' \in \mathbb{Z}_r, s' \in \mathbb{Z}_s\}$ is a super vertex-cut in $G \times K_n$ with $|S| = n + 2(n-1) + (m-3) = 3n + m - 5$.

Subcase 2. Let there exist some vertices $x_p \in N_{G-\{u_2,u_3\}}(u_1)$ and $y_q \in N_{G-\{u_1,u_3\}}(u_2)$ such that $x_p \sim y_q$. Since $\kappa'(G) = \infty$ if $x_p \sim y_q$ and $x_{p'} \sim y_{q'}$ then $x_p \sim y_{q'}$ or $x_{p'} \sim y_q$. Let h be the number of vertices $x_{r'}$ with $\deg(x_{r'}) \geq 2$ and l be the number of vertices $y_{s'}$ with $\deg(y_{s'}) \geq 2$. Without loss of generality let $l \leq h$, and $\{y_1, y_2, \cdots, y_l\} \subset N_{G-\{u_1,u_3\}}(u_2)$ be such that $\deg(y_q) \geq 2$ for $q \in \mathbb{Z}_l$. Then $S = \{(u_3, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(u_1, v_j), (u_2, v_j) \mid j \in \mathbb{Z}_n - \{1\}\} \cup \{(y_q, v_i) \mid q \in \mathbb{Z}_l, i \in \mathbb{Z}_n - \{1\}\} \cup \{(y_{s'}, v_1) \mid l < s' \leq s\} \cup \{(x_{r'}, v_1) \mid r' \in \mathbb{Z}_r\}$ is a super vertex-cut in $G \times K_n$ with $|S| = n + 2(n-1) + l(n-1) + (s-l) + r = 3n + m - 5 + l(n-2)$. If $l = s$ then (u_2, v_1) is an isolated vertex in $(G \times K_n) - S$. So $l < s$. Now $l \leq m - 5$ implies that $|S| \leq n(m-2) - m + 5$.

Now, let $\deg(x_{r'}) \geq 2$ and $\deg(y_{s'}) \geq 2$ for every $x_{r'} \in N_{G-\{u_2,u_3\}}(u_1)$, $y_{s'} \in N_{G-\{u_1,u_3\}}(u_2)$. We choose a constant $l \neq 1$ of $\mathbb{Z}_n$. Hence $S = \{(u_3, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(u_1, v_i), (u_2, v_i) \mid i \in \mathbb{Z}_n - \{1\}\} \cup \{(x_{r'}, v_i), (y_{s'}, v_i) \mid i \in \mathbb{Z}_n - l\}$ is a super vertex-cut in $G \times K_n$ with $|S| = n + 2(n-1) + (m-3)(n-1) = m(n-1) + 1$.

Now, suppose that G contains t cycles which have common edges. We consider the following cases:

Case 1. Let G be isomorphic to t triangles which have a common edge $\{a, b\}$. Also, let the other vertices be the set $\{\alpha_1, \alpha_2, \cdots, \alpha_{m-2}\}$. Now $S = \{(\alpha_j, v_i), (a, v_i) \mid j \in \mathbb{Z}_{m-2}, i \in \mathbb{Z}_n - \{1, 2\}\} \cup \{(b, v_i) \mid i \in \mathbb{Z}_n\}$ is a super vertex-cut in $G \times K_n$ with $|S| = mn - 2m + 2$. Also let G be isomorphic to t triangles and a square which have a common edge $\{a, b\}$. Let $\{a, b, \alpha_1, \alpha_2\}$ be vertices of square and $\{\alpha_3, \cdots, \alpha_{m-2}\}$ be other vertices. Again S is a super vertex-cut in $G \times K_n$ with $|S| = mn - 2m + 2$. Furthermore, if G has more than one square with common edge $\{a, b\}$ then it is clear that $\kappa'(G) < \infty$, a contradiction. Assume that G contains more than one square and there are some edges between the vertices of squares such that $\kappa'(G) = \infty$ then we have two subcases:

Subcase 1. If the vertices of square are such that there is no pentagon then S is as above.

Subcase 2. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ be the vertices of triangles and $\{\alpha_{s+1}, \alpha_{s+2}, \cdots, \alpha_{m-2}\}$ be the vertices of squares. We have to delete some vertices for removing pentagons. Let $\{\alpha_{s+1}, \alpha_{s+2}, \cdots, \alpha_{s+l}\}$ be the minimum vertices which deleting them removes pentagons. Hence $S = \{(b, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(a, v_i) \mid i \in \mathbb{Z}_n - \{1,2\}\} \cup \{(\alpha_j, v_i) \mid j \in \mathbb{Z}_s, i \in \mathbb{Z}_n - \{1,2\}\} \cup \{(\alpha_j, v_i) \mid s+1 \leq j \leq s+l, i \in \mathbb{Z}_n\} \cup \{(\alpha_j, v_i) \mid s+l+1 \leq j \leq m-2, i \in \mathbb{Z}_n - \{1,2\}\}\}$ is a super vertex-cut in $G \times K_n$ with $|S| = n + (n-2) + s(n-2) + ln + (m-2-s-l-1+1)(n-2) = mn - 2m + 2 + 2l$. Thus $l \leq m-5$ implies that $|S| \leq mn - 8$.

Moreover, if G is isomorphic to Figure.1, then $S = \{(a, v_i), (b, v_i) \mid i \in \mathbb{Z}_n - \{1\}\} \cup \{(\beta_j, v_1), (\gamma_r, v_1) \mid j \in \mathbb{Z}_{\acute{f}}, r \in \mathbb{Z}_{\grave{f}}\} \cup \{(\alpha_e, v_i) \mid e \in \mathbb{Z}_f, i \in \mathbb{Z}_n\}$ is a super vertex-cut in $G \times K_n$ with $|S| = 2(n-1) + \acute{f} + \grave{f} + fn$. Now, $\acute{f} + \grave{f} + f = m-2$ and $f \leq m-4$ implies that $|S| \leq 2(n-1) + m - 2 + f(n-1) \leq n(m-2)$.

Also, if G is isomorphic to Figure.2, then $S = \{(a, v_i), (b, v_i), (\alpha_1, v_i) \mid i \in \mathbb{Z}_n - \{1\}\} \cup \{(\beta_j, v_1), (\gamma_r, v_1), (\lambda_w, v_1) \mid j \in \mathbb{Z}_{\acute{f}}, r \in \mathbb{Z}_{\grave{f}}, w \in \mathbb{Z}_{\dot{f}}\} \cup \{(\alpha_e, v_i) \mid 2 \leq e \leq f, i \in \mathbb{Z}_n\}$ is a super vertex-cut in $G \times K_n$. Now $\acute{f} + \grave{f} + \dot{f} + f = m-2$ and $f - 1 \leq m-6$ implies that $|S| \leq 3(n-1) + \acute{f} + \grave{f} + \dot{f} + (f-1)n \leq n(m-3)$.

Case 2. Let G contains three cycles C_1, C_2 and C_3 with $\mathrm{V}(G) = \{u_1, u_2, u_3, \cdots, u_m\}$. Let $V(C_1) = \{u_1, u_2, u_3\}$, $V(C_2) = \{u_2, u_3, u_4\}$ and $V(C_3) = \{u_2, u_4, u_5\}$. If $\deg(u_1) = \deg(u_5) = 2$, $\deg(u_2) = 4$ and $\deg(u_3) = \deg(u_4) = 3$ then $S = \cup_{i=3}^{n} S_i \cup \{(u_2, v_1), (u_2, v_2)\}$ is a super vertex-cut in $G \times K_n$ with $|S| = m(n-2) + 2$. Now suppose that $\deg(u_2) \geq 5$ and either $\deg(u_3) \geq 4$ or $\deg(u_4) \geq 4$. Let $N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) = \{x_{r'} \mid r' \in \mathbb{Z}_r\}$, $N_{G-\{u_1,u_2,u_4\}}(u_3) = \{y_{s'} \mid s' \in \mathbb{Z}_s\}$ and $N_{G-\{u_2,u_3,u_5\}}(u_4) = \{w_{t'} \mid t' \in \mathbb{Z}_t\}$ where $r + s + t = m - 5$. Let $\deg(x_{r'}) = \deg(y_{s'}) = \deg(w_{t'}) = 1$, for every $r' \in \mathbb{Z}_r$, $s' \in \mathbb{Z}_s$ and $t' \in \mathbb{Z}_t$. So $S = \{(u_j, v_i) \mid 2 \leq j \leq 4, i \in \mathbb{Z}_n - \{1\}\} \cup \{(u_1, v_1)\} \cup \{(u_5, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(x_{r'}, v_1), (y_{s'}, v_1), (w_{t'}, v_1) \mid r' \in \mathbb{Z}_r, s' \in \mathbb{Z}_s, t' \in \mathbb{Z}_t\}$ is a super vertex-cut in $G \times K_n$ with $|S| = 3(n-1) + 1 + n + (m-5) = 4n - 7 + m$. Also if $x_{r'_i} \sim x_{r'_j}$ or $y_{s'_i} \sim y_{s'_j}$ or $w_{t'_i} \sim w_{t'_j}$ or $x_{r'_i} \sim y_{s'_j}$ or $x_{r'_i} \sim w_{t'_j}$ or $y_{s'_i} \sim w_{t'_j}$ for some i, j then $\kappa'(G) < \infty$. Moreover if $u_3 \sim w_{t'_i}$ or $u_4 \sim y_{s'_i}$ then $S = \{(u_j, v_i) \mid 2 \leq j \leq 4, i \in \mathbb{Z}_n - \{1\}\} \cup \{(u_1, v_i), (u_5, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(x_{r'}, v_1), (y_{s'}, v_1), (w_{t'}, v_1) \mid r' \in \mathbb{Z}_r, s' \in \mathbb{Z}_s, t' \in \mathbb{Z}_t\}$ is a super vertex-cut in $G \times K_n$ with $|S| = 3(n-1) + 2n + (m-5) = 5n - 8 + m$. Now let $x_{r'_i} \sim u_3$ or $x_{r'_j} \sim u_4$ for some i, j. Let $A = \{x \in N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) \mid x \sim u_3 \text{ and } x \sim u_4\}$, $B = \{x \in N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) \mid x \sim u_3 \text{ and } xu_4 \notin E(G)\}$, $C = \{x \in N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) \mid xu_3 \notin E(G) \text{ and } x \sim u_4\}$, $D = \{x \in N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) \mid x \sim \acute{x}$ for some $\acute{x}$; $(\acute{x} \in N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) \cap N_{G-\{u_1,u_2,u_4\}}(u_3)$ *and* $\acute{x} \notin N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) \cap N_{G-\{u_2,u_3,u_5\}}(u_4)$ or $(\acute{x} \in N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) \cap N_{G-\{u_2,u_3,u_5\}}(u_4)$ and $\acute{x} \notin N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) \cap N_{G-\{u_1,u_2,u_4\}}(u_3)$ and $F = \{x \in N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) \mid x \notin A \cup B \cup C \cup D\}$ with $|A| = l$, $|B \cup C| = \acute{l}$, $|D| = \dot{l}$ and $|F| = \grave{l}$. Then $S = \{(u_3, v_i), (u_4, v_i) \mid i \in \mathbb{Z}_n - \{1\}\} \cup \{(u_2, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(u_1, v_1), (u_5, v_1)\} \cup \{(x, v_i) \mid x \in A \text{ } or \text{ } x \in F, i \in \mathbb{Z}_n\} \cup \{(x, v_1) \mid x \in B \cup C\} \cup \{(y_{\acute{s}}, v_1), (w_{\acute{t}}, v_1)\}$ is a super vertex-cut in $G \times K_n$ with $|S| = 2(n-1) + n + 2 + (l + \grave{l})n + (\acute{l}) + (m - 5 - r)$. Now, $l + \acute{l} + \grave{l} = r - \dot{l}$, $\dot{l} \leq m - 7$ and $l + \grave{l} \leq m - 5$ implies that $|S| \leq n(m-2) - m + 7$.

Moreover, if there are vertices $h \in N_{G-\{u_2,u_3\}}(u_1) - N_{G-\{u_2,u_4\}}(u_5)$ or $\acute{h} \in N_{G-\{u_2,u_4\}}(u_5) - N_{G-\{u_2,u_3\}}(u_1)$ then $\kappa^{'}(G) < \infty$, a contradiction. Let $\{h_1, \cdots, h_f\} = N_{G-\{u_2,u_3\}}(u_1) \cap N_{G-\{u_2,u_4\}}(u_5)$. If for some i, j, $h_i \sim h_j$ or $\deg(u_3) \geq 4$ or $\deg(u_4) \geq 4$ then $\kappa^{'}(G) < \infty$, a contradiction. Hence let for every i, j, $h_i h_j \notin E(G)$, $\deg(u_3) = \deg(u_4) = 3$ and $N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) = \{x_{r'} \mid r^{'} \in \mathbb{Z}_r\}$. So $S = \{(u_3, v_i), (u_4, v_i)) \mid i \in \mathbb{Z}_n\} \cup \{(u_1, v_i), (u_2, v_i), (u_5, v_i) \mid i \in \mathbb{Z}_n - \{1\}\} \cup \{(h_j, v_1), (x_{r'}, v_1) \mid j \in \mathbb{Z}_f, \acute{r} \in \mathbb{Z}_r\}$ is a super vertex-cut in $G \times K_n$ with $|S| = 2n + 3(n-1) + (m-5) = 5n + m - 8$.
Now, let $\deg(u_2) = 4$, $\deg(u_3) = 3$, $\deg(u_4) = 3$ and $u_1 \sim u_5$. The graph is W_4 and has four triangles. In this case, $S = \{(u_2, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(u_j, v_i) \mid j \in \mathbb{Z}_5 - \{2\}, i \in \mathbb{Z}_n - \{1,2\}\}$ is a super vertex-cut in $G \times K_n$ with $|S| = n + 4(n-2) = 5n - 8$. Furthermore, let $u_1 \sim u_5$, $\deg(u_2) \geq 4$, $\deg(u_3) \geq 3$ and $\deg(u_4) \geq 3$. Let $N_{G-\{u_1,u_3,u_4,u_5\}}(u_2) = \{x_{r'} \mid r^{'} \in \mathbb{Z}_r\}$, $N_{G-\{u_1,u_2,u_4\}}(u_3) = \{y_{s'} \mid s^{'} \in \mathbb{Z}_s\}$ and $N_{G-\{u_2,u_3,u_5\}}(u_4) = \{w_{t'} \mid t^{'} \in \mathbb{Z}_t\}$ where $r + s + t = m - 5$, and $\deg(x_{r'}) = \deg(y_{s'}) = \deg(w_{t'}) = 1$ for each $r^{'}, s^{'}, t^{'}$. Then $S = \{(u_j, v_i) \mid 2 \leq j \leq 4, i \in \mathbb{Z}_n - \{1\}\} \cup \{(u_5, v_i), (u_1, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(x_{r'}, v_1), (y_{s'}, v_1), (w_{t'}, v_1) \mid r^{'} \in \mathbb{Z}_r, s^{'} \in \mathbb{Z}_s, t^{'} \in \mathbb{Z}_t\}$ is a super vertex-cut in $G \times K_n$ with $|S| = 3(n-1) + 2n + (m-5) = 5n - 8 + m$. Now if $x_{r_i^{'}} \sim u_3$ or $x_{r_j^{'}} \sim u_4$ for some i, j, then by using A, B, C, D and F as above, and putting $\{(u_1, v_i), (u_5, v_i) \mid i \in \mathbb{Z}_n\}$ in S we get S a super vertex-cut in $G \times K_n$ with $|S| \leq n(m-2) - m + 7 + 2n - 2 = m(n-1) + 5$.

Now let G be the graph W_5 with vertices $\{u_i\}_{i \in \mathbb{Z}_6}$ where $u_1 \sim u_3 \sim u_4 \sim u_5 \sim u_6 \sim u_1$ and u_2 is adjacent to all others. Hence by putting $\{(u_2, v_i) \mid i \in \mathbb{Z}_n\}$ in S and using Theorem 2.2, we get $|S| = \min\{4n, 6n - 8\}$. Now let $\deg(u_2) \geq 6$, $N_{G-\{u_1,u_3,u_4,u_5,u_6\}}(u_2) = \{x_{r'} \mid r^{'} \in \mathbb{Z}_r\}$. If for some i, j, $x_{r_i^{'}} \sim x_{r_j^{'}}$ or $x_{r_i^{'}} \sim u_s$ for $s \in \mathbb{Z}_n - \{2\}$, then $\kappa^{'}(G) < \infty$, a contradiction. Let for all $\acute{r} \in \mathbb{Z}_r$, $\deg(x_{r'}) = 1$. Then $S = \{(u_1, v_i), (u_2, v_i)(u_5, v_i) \mid i \in \mathbb{Z}_n - \{1,2\}\} \cup \{(u_3, v_i), (u_4, v_i), (u_6, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(x_{r'}, v_i) \mid r^{'} \in \mathbb{Z}_r, i \in \mathbb{Z}_n - \{1,2\}\}$ is a super vertex-cut in $G \times K_n$ with $|S| = m(n-2) + 6$. Also let $u_1 \sim u_4$ or $u_1 \sim u_5$ or $u_3 \sim u_5$ or $u_3 \sim u_6$ or $u_4 \sim u_6$ where $\deg(u_2) = 5$. Since G is not complete we may assume that $\deg(u_1) = \deg(u_4) = 4$ and $u_1 u_4 \notin E(G)$. Thus $S = \{(u_2, v_i), (u_5, v_i), (u_6, v_i) \mid i \in \mathbb{Z}_n\} \cup \{(u_1, v_i), (u_3, v_i), (u_4, v_i) \mid i \in \mathbb{Z}_n - \{1,2\}\}$ is a super vertex-cut in $G \times K_n$ with $|S| = 6n - 6$. Similarly, for any non complete graph G with vertices $\{x_1, \cdots, x_m\}$ and $\text{girth}(G) = 3$, if $x_1 x_3 \notin E(G)$ and $x_1 \sim x_2 \sim x_3$, then $S = \{(x_j, v_i) \mid j \in \mathbb{Z}_m - \{1,2,3\}, i \in \mathbb{Z}_n\} \cup \{(x_j, v_i) \mid j \in \mathbb{Z}_3, i \in \mathbb{Z}_n - \{1,2\}\}$ is a super vertex-cut in $G \times K_n$ with $|S| = (m-3)n + 3(n-2) = mn - 6$. Also, $\kappa^{'}(W_n) < \infty$ where $n \geq 6$. If a graph G includes W_n with more than $2n$ vertices where $\kappa^{'}(G) = \infty$ then by above argument we see that the cardinality of a super vertex-cut is $mn - 6$.
Therefore, if $\text{girth}(G) = 3$ and $\kappa^{'}(G) = \infty$ then, $\kappa^{'}(G \times K_n) \leq \min\{mn - 6, m(n-1) + 5, 5n + m - 8\}$. ∎

Conclusion In this article we provide bounds for the super connectivity of the direct product of an arbitrary graph and the complete graph K_n. Also, we show that if G is a non-complete graph with $\text{girth}(G) = 3$ and $\kappa^{'}(G) = \infty$, then $\kappa^{'}(G \times K_n) \leq \min\{mn - 6, m(n-1) + 5, 5n + m - 8\}$, where $|V(G)| = m$.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Elsevier, New York, 1967.

[2] G.B. Ekinci, J.B. Gauci, The super-connectivity of odd graphs and of their kronecker double cover, Rairo-Oper. Res. 55 (2) (2021), 561–566.

[3] G.B. Ekinci, A. Kirlangiç, Super connectivity of kronecker product of complete bipartite graphs and complete graphs, Discrete Math. 339 (2016), 1950–1953.

[4] G.B. Ekinci, A. Kirlangiç, The super edge connectivity of Kronecker product graphs, Rairo-Oper. Res. 52 (2) (2018), 561–566.

[5] S. Ghozati, A finite automata approach to modeling the cross product of interconnection networks, Math. Compute. Modelling 30 (7) (1999), 185–200.

[6] R. Guji, E. Vumar, A note on the connectivity of kronecker products of graphs, Appl. Math. Lett. 22 (9) (2009), 1360–1363.

[7] L. Guo, X. Guo. Super connectivity of Kronecker products of some graphs, Ars. Combin. 123 (2015) 65–73.

[8] L. Guo, C. Qin, X. Guo, Super connectivity of kronecker products of graphs, Inform. Process. Lett. 110 (16) (2010), 659–661.

[9] L. Guo, W. Yang, X. Guo, Super-connectivity of kronecker products of split graphs, powers of cycles, powers of paths and complete graphs, Appl. Math. Lett. 26 (1) (2013), 120–123.

[10] Kh, Kamyab, M. Ghasemi, R. Varmazyar , Super connectivity of lexicographic product graphs, (accepted to Ars Combinatoria) arXiv:2009.04831[math.GR].

[11] R. Lammprey, B. Barnes, Products of graphs and applications, Model. Simul. 5 (1974), 1119–1123.

[12] J. Leskovec, D. Chakrabarti, J. Kleinberg, C. Faloutsos, Z. Ghahramani, Kronecker graphs: an approach to modeling networks, J. Mach. Learn. Res. 11 (2010), 985–1042.

[13] M. Lü, C. Wu, G.L. Chen, C. Lv, On super connectivity of cartesian product graphs, Networks 52 (2008) 78–87.

[14] M. Ma, G. Liu, J.M. Xu, The super connectivity of augmented cubes, Inform. Process. Lett. 106 (2) (2008), 59–63.

[15] D.J. Miller, The categorical product of graphs, Canad. J. Math. 20 (6) (1968), 1511-1521.

[16] F. Soliemany, M. Ghasemi, R. Varmazyar, Super connectivity of a family of direct product graphs, Int. J. Comput. Math. Comput. Syst. Theory. 7 (1) (2021) 1–5.

[17] P.M. Weichsel, The kronecker product of graphs, Proc. Amer. Math. Soc. 13 (1) (1962), 47–52.

[18] J.M. Xu, Topological structure and analysis of interconnection networks, Kluwer Academic Publishers, Dordrecht, 2001.

[19] J.M. Xu, M. Lü, M. Ma, A. Hellwig, Super connectivity of line graphs, Inform. Process. Lett. 94 (4) (2005) 191–195.

[20] C.S. Yang, J.F. Wang, J.Y. Lee, F.T. Boesch, Graph theoretic reliable analysis for the Boolean n-cube networks, IEEE Trans Circuits Syst 35 (1988), 1175–1179.

[21] C.S. Yang, J.F. Wang, J.Y. Lee, F.T. Boesch, The number of spanning trees of the regular networks, Int J Comput Math 23 (1988), 185–200.

[22] J.X. Zhou, Super connectivity of direct product of graphs, Ars Math. Contemp. 8(2) (2015), 235–244.

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