Isolated toughness for path factors in networks

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Abstract

Let $\mathcal{H}$ be a set of connected graphs. Then an $\mathcal{H}$-factor is a spanning subgraph of G, whose every connected component is isomorphic to a member of the set $\mathcal{H}$. An $\mathcal{H}$-factor is called a path factor if every member of the set $\mathcal{H}$ is a path. Let $k \geq 2$ be an integer. By a $P_{\geq k}$-factor we mean a path factor in which each component path admits at least k vertices. A graph G is called a $(P_{\geq k}, n)$-factor-critical covered graph if for any $W \subseteq V(G)$ with $|W| = n$ and any $e \in E(G-W)$, $G-W$ has a $P_{\geq k}$-factor covering e. In this article, we verify that (i) an $(n+\lambda+2)$-connected graph G is a $(P_{\geq 2}, n)$-factor-critical covered graph if its isolated toughness $I(G) > \frac{n+\lambda+2}{2\lambda+3}$, where n and λ are two nonnegative integers; (ii) an $(n+\lambda+2)$-connected graph G is a $(P_{\geq 3}, n)$-factor-critical covered graph if its isolated toughness $I(G) > \frac{n+3\lambda+5}{2\lambda+3}$, where n and λ be two nonnegative integers.

Keywords: graph; isolated toughness; $P_{\geq k}$-factor; $P_{\geq k}$-factor covered graph; $(P_{\geq k}, n)$-factor-critical covered graph.

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1 Introduction

We may model real-world networks by graphs. The vertices of the graph stand for the nodes of the network, and the edges of the graph act as the links between the nodes in the network. Henceforth, we replace "network" by the term "graph". Thus we may use several graphic parameters to characterize the robustness and vulnerability of the network, for instance, isolated toughness, independence number and minimum degree, and so on. In data transmission networks, the data transmission between two sites of a network goes through a path between two corresponding vertices of a corresponding graph. Therefore, the availability of data transmission in the network is equal to the existence of path factor in the corresponding graph which is generated by the network. When some nodes are damaged and a special channel is assigned, the possibility of data transmission in a data transmission network is equivalent to the existence of path factor critical covered graph. Research on the existence of path factors or path factor critical covered graphs under specific network structures can help scientists to design and construct networks with high data transmission rates. In this article, we study the existence of path factor critical

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covered graphs which play a key role in investigating data transmissions of data transmission networks. We find that there is strong essential connection between isolated toughness and the existence of path factor critical covered graphs, and hence investigations on isolated toughness, which play an irreplaceable role in the vulnerability of the network and the feasibility of data transmission, can yield theoretical guidance to meet data transmission and network security requirements.

In this article, we discuss only finite undirected graphs without loops or multiple edges. We denote a graph by $G = (V(G), E(G))$, where $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G. Let $i(G)$ denote the number of isolated vertices of G, and let $\omega(G)$ denote the number of connected components of G. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G. Let X be a subset of $V(G)$. We call that X is independent if no two elements in X are adjacent, and we denote by $G[V(G) \setminus X]$ the subgraph of G induced by $V(G) \setminus X$. Yang, Ma and Liu [13] defined a graphic parameter, isolated toughness of G, denoted by $I(G)$, namely,

$$I(G) = \min \left\{ \frac{|X|}{i(G-X)} : X \subseteq V(G), i(G-X) \geq 2 \right\}$$

if G is not a complete graph; otherwise, $I(G) = +\infty$. Write K_n and P_n for the complete graph and the path of order n, respectively.

Let $\mathcal{H}$ be a set of connected graphs. Then an $\mathcal{H}$-factor is a spanning subgraph of G, whose every connected component is isomorphic to a member of the set $\mathcal{H}$. An $\mathcal{H}$-factor is called a path factor if every member of the set $\mathcal{H}$ is a path. Let $k \geq 2$ be an integer. By a $P_{\geq k}$-factor we mean a path factor in which each component path admits at least k vertices. A graph G is called a $P_{\geq k}$-factor covered graph if for any $e \in E(G)$, G has a $P_{\geq k}$-factor containing e. A graph G is called a $(P_{\geq k}, n)$-factor-critical covered graph if for any $Q \subseteq V(G)$ with $|Q| = n$, $G - Q$ is a $P_{\geq k}$-factor covered graph.

To characterize a graph with a $P_{\geq 3}$-factor, Kaneko [5] introduced the concept of a sun. A graph R is called a factor-critical graph if for any $x \in V(R)$, $R - x$ has a perfect matching. Assume that R is a factor-critical graph with vertex set $V(R) = \{x_1, x_2, \cdots, x_n\}$. By adding new vertices $y_1, y_2, \cdots, y_n$ together with new edges $x_1y_1, x_2y_2, \cdots, x_ny_n$ to R, we acquire a new graph, which is called a sun. In particular, K_1 and K_2 are also called suns. A sun with at least six vertices is called a big sun. A component of $G - X$ is called a sun component if it is isomorphic to a sun. Let $sun(G - X)$ be the number of sun components in $G - X$.

A criterion for a graph to admit a $P_{\geq 2}$-factor was derived by Las Vergnas [9].

Theorem 1.1 ([9]). A graph G has a $P_{\geq 2}$-factor if and only if

$$i(G - X) \leq 2|X|$$

for all $X \subseteq V(G)$.

A characterization for a graph with a $P_{\geq 3}$-factor was provided by Kaneko [5].

Theorem 1.2 ([5]). A graph G has a $P_{\geq 3}$-factor if and only if

$$sun(G - X) \leq 2|X|$$

for all $X \subseteq V(G)$.

Zhang and Zhou [15] extended Theorems 1.1 and 1.2, and got two characterizations for graphs to be $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs.

Theorem 1.3 ([15]). A connected graph G is a $P_{\geq 2}$-factor covered graph if and only if

$$i(G-X) \leq 2|X| - \varepsilon_1(X)$$

for all $X \subseteq V(G)$, where $\varepsilon_1(X)$ is defined by

$$\varepsilon_1(X) = \begin{cases} 2, & if\ X\ is\ not\ an\ independent\ set; \\ 1, & if\ X\ is\ a\ nonempty\ independent\ set,\ and\ G-X\ admits \\ & a\ nontrivial\ component; \\ 0, & otherwise. \end{cases}$$

Theorem 1.4 ([15]). A connected graph G is a $P_{\geq 3}$-factor covered graph if and only if

$$sun(G-X) \leq 2|X| - \varepsilon_2(X)$$

for all $X \subseteq V(G)$, where $\varepsilon_2(X)$ is defined by

$$\varepsilon_2(X) = \begin{cases} 2, & if\ X\ is\ not\ an\ independent\ set; \\ 1, & if\ X\ is\ a\ nonempty\ independent\ set,\ and\ G-X\ admits \\ & a\ non-sun\ component; \\ 0, & otherwise. \end{cases}$$

Kano, Lu and Yu [7] claimed that a graph has a $P_{\geq 3}$-factor if $i(G-X) \leq \frac{2}{3}|X|$ for any $X \subseteq V(G)$. Kawarabayashi, Matsuda, Oda and Ota [8] showed that a 2-connected cubic graph with at least six vertices admits a $P_{\geq 6}$-factor. Kano, Lee and Suzuki [6] verified that a connected cubic bipartite graph with at least eight vertices has a $P_{\geq 8}$-factor. Zhou [19], Zhou, Bian and Pan [20], Zhou, Sun and Liu [23], Gao and Wang [3], Zhou, Sun and Bian [22] discussed the existence of $P_{\geq 2}$-factors and $P_{\geq 3}$-factors with given properties in graphs. Zhou, Wu and Xu [25] posed two sufficient conditions for a graph to be a $(P_{\geq 3}, n)$-factor-critical covered graph. Gao, Wang and Chen [4] showed a binding number condition for a graph to be a $(P_{\geq 3}, n)$-factor-critical covered graph. Zhou, Wu and Bian [24] obtained a result on the existence of $(P_{\geq 3}, n)$-factor-critical covered graphs. Some recent advances on graph factors can be found in Zhou, Liu and Xu [21], Zhou [16–18], Zhou, Zhang and Xu [26], Wang and Zhang [10–12], Yuan and Hao [14], Anstee and Nam [1], Egawa, Kano and Yokota [2]. In this article, we study the existence of $(P_{\geq 2}, n)$-factor-critical covered graphs and $(P_{\geq 3}, n)$-factor-critical covered graphs, and derive the following two results.

Theorem 1.5. Let n and λ be two nonnegative integers. Then an $(n+\lambda+2)$-connected graph G is a $(P_{\geq 2}, n)$-factor-critical covered graph if its isolated toughness $I(G) > \frac{n+\lambda+2}{2\lambda+3}$.

Theorem 1.6. Let n and λ be two nonnegative integers. Then an $(n+\lambda+2)$-connected graph G is a $(P_{\geq 3}, n)$-factor-critical covered graph if its isolated toughness $I(G) > \frac{n+3\lambda+5}{2\lambda+3}$.

2 The proof of Theorem 1.5

Proof of Theorem 1.5. If G is a complete graph, then it is obvious that G is a $(P_{\geq 2}, n)$-factor-critical covered graph by G being $(n+\lambda+2)$-connected. In what follows, we consider that G is not a complete graph.

Let $G' = G - W$ for any $W \subseteq V(G)$ with $|W| = n$. To prove Theorem 1.5, it suffices to verify that G' is a $P_{\geq 2}$-factor covered graph. By means of contrary, we assume that G' is not a $P_{\geq 2}$-factor covered graph. Then it follows from Theorem 1.3 that

$$i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \tag{2.1}$$

for some vertex subset X of G'.

Claim 1. $|X| \geq \lambda + 2$.

Proof. Let $|X| \leq \lambda + 1$. Note that G is $(n + \lambda + 2)$-connected and $G' = G - W$. We see that $G' - X$ is connected, and so $i(G' - X) = 0$. Combining this with (2.1) and $\varepsilon(X) \leq |X|$, we derive

$$0 = i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq |X| + 1 \geq 1,$$

which is a contradiction. This completes the proof of Claim 1. □

By virtue of (2.1), $\varepsilon_1(X) \leq 2$ and Claim 1, we admit

$$i(G - W - X) = i(G' - X) \geq 2|X| - \varepsilon_1(X) + 1 \geq 2|X| - 1 \geq 2(\lambda + 2) - 1 = 2\lambda + 3 > 2. \tag{2.2}$$

Using (2.2), $\varepsilon_1(X) \leq 2$, Claim 1 and the definition of $I(G)$, we obtain

$$\begin{aligned} I(G) &\leq \frac{|W \cup X|}{i(G - W - X)} \leq \frac{n + |X|}{2|X| - \varepsilon_1(X) + 1} \\ &\leq \frac{n + |X|}{2|X| - 1} = \frac{1}{2} + \frac{n + \frac{1}{2}}{2|X| - 1} \\ &\leq \frac{1}{2} + \frac{n + \frac{1}{2}}{2(\lambda + 2) - 1} = \frac{n + \lambda + 2}{2\lambda + 3}, \end{aligned}$$

which contradicts that $I(G) > \frac{n+\lambda+2}{2\lambda+3}$. Theorem 1.5 is verified. □

Remark 2.1. Next, we claim that the condition $I(G) > \frac{n+\lambda+2}{2\lambda+3}$ in Theorem 1.5 is sharp. We construct a graph $G = K_{n+\lambda+2} \vee ((2\lambda + 3)K_1)$, where n and λ are two nonnegative integers, $\vee$ means "join". Obviously, G is $(n + \lambda + 2)$-connected and $I(G) = \frac{n+\lambda+2}{2\lambda+3}$. For $W \subseteq V(K_{n+\lambda+2})$ with $|W| = n$, let $G' = G - W = K_{\lambda+2} \vee ((2\lambda + 3)K_1)$. Select $X = V(K_{\lambda+2})$ in G'. Then $\varepsilon_1(X) = 2$, and we obtain

$$i(G' - X) = 2\lambda + 3 > 2\lambda + 2 = 2(\lambda + 2) - 2 = 2|X| - \varepsilon_1(X).$$

In terms of Theorem 1.3, G' is not a $P_{\geq 2}$-factor covered graph, and so G is not a $(p_{\geq 2}, n)$-factor-critical covered graph.

Remark 2.2. In what follows, we claim that the condition $(n + \lambda + 2)$-connectivity in Theorem 1.5 is best possible. We construct a graph $G = K_{n+\lambda+1} \vee ((2\lambda + 1)K_1)$, where $n \geq 0$ and $\lambda \geq 1$ are two integers, $\vee$ means "join". Clearly, $I(G) = \frac{n+\lambda+1}{2\lambda+1} > \frac{n+\lambda+2}{2\lambda+3}$ and G is $(n + \lambda + 1)$-connected. Let $G' = G - W = K_{\lambda+1} \vee ((2\lambda + 1)K_1)$, where $W \subseteq V(K_{n+\lambda+1})$ with $|W| = n$. Choose $X = V(K_{\lambda+1})$ in G'. Then $\varepsilon_1(X) = 2$, and we admit

$$i(G' - X) = 2\lambda + 1 > 2\lambda = 2(\lambda + 1) - 2 = 2|X| - \varepsilon_1(X).$$

By means of Theorem 1.3, G' is not a $P_{\geq 2}$-factor covered graph, and so G is not a $(p_{\geq 2}, n)$-factor-critical covered graph.

3 The proof of Theorem 1.6

Proof of Theorem 1.6. If G is a complete graph, then it is clear that G is a $(P_{\geq 3}, n)$-factor-critical covered graph by G being $(n+\lambda+2)$-connected. In what follows, we consider that G is not a complete graph.

Let $G' = G - W$ for any $W \subseteq V(G)$ with $|W| = n$. To verify Theorem 1.6, it suffices to justify that G' is a $P_{\geq 3}$-factor covered graph. By means of contrary, we assume that G' is not a $P_{\geq 3}$-factor covered graph. Then by Theorem 1.4, we have

$$sun(G' - X) \geq 2|X| - \varepsilon_2(X) + 1 \tag{3.1}$$

for some vertex subset X of G'.

Next, we shall discuss two cases by the value of $|X|$, and derive a contradiction in each case.

Case 1. $0 \leq |X| \leq \lambda + 1$.

Since $G' = G - W$ and G is $(n+\lambda+2)$-connected, G' is $(\lambda+2)$-connected. Combining this with the definition of sun component, we know that $sun(G') = 0$. If $|X| = 0$, then by (3.1) and $\varepsilon_2(X) \leq |X|$, we derive

$$0 = sun(G') = sun(G' - X) \geq 2|X| - \varepsilon_2(X) + 1 \geq |X| + 1 = 1,$$

which is a contradiction.

If $1 \leq |X| \leq \lambda + 1$, then we have

$$\omega(G' - X) = 1 \tag{3.2}$$

by G' being $(\lambda+2)$-connected. By virtue of (3.1), (3.2) and $\varepsilon_2(X) \leq |X|$, we get

$$1 = \omega(G' - X) \geq sun(G' - X) \geq 2|X| - \varepsilon_2(X) + 1 \geq |X| + 1 \geq 2,$$

which is a contradiction.

Case 2. $|X| \geq \lambda + 2$.

In this case, we assume that there exist a isolated vertices, b K_2's and c big sun components $Q_1, Q_2, \cdots, Q_c$, where $|V(Q_i)| \geq 6$ for $1 \leq i \leq c$, in $G' - X$, and so

$$sun(G' - X) = a + b + c. \tag{3.3}$$

We choose one vertex from every K_2 component of $G' - X$, and use Y to denote the set of such vertices. Let R_i be the factor-critical subgraph of Q_i for $1 \leq i \leq c$. Obviously, we have $|Y| = b$ and $i(Q_i - V(R_i)) = |V(R_i)|$. Thus, we derive

$$\begin{aligned}
&i(G - (W \cup X \cup Y \cup V(R_1) \cup \cdots \cup V(R_c)))\\
&= i(G' - (X \cup Y \cup V(R_1) \cup \cdots \cup V(R_c)))\\
&= a + b + i(Q_1 - V(R_1)) + \cdots + i(Q_c - V(R_c))\\
&= a + b + |V(R_1)| + \cdots + |V(R_c)|\\
&= a + b + \sum_{i=1}^{c} |V(R_i)|.
\end{aligned} \tag{3.4}$$

It follows from (3.1), (3.3), (3.4), $\varepsilon_2(X) \leq 2$ and $|V(R_i)| \geq 3$ that

$$i(G - (W \cup X \cup Y \cup V(R_1) \cup \cdots \cup V(R_c)))$$

$$
\begin{aligned}
&= a + b + \sum_{i=1}^{c} |V(R_i)| \\
&\geq a + b + 3c \geq a + b + c = sun(G' - X) \\
&\geq 2|X| - \varepsilon_2(X) + 1 \geq 2|X| - 1 \\
&\geq 2(\lambda + 2) - 1 = 2\lambda + 3 \geq 3. \qquad (3.5)
\end{aligned}
$$

In terms of (3.5), $I(G) > \frac{n+3\lambda+5}{2\lambda+3}$ and the definition of $I(G)$, we deduce

$$
\begin{aligned}
\frac{n + 3\lambda + 5}{2\lambda + 3} < I(G) &\leq \frac{|W \cup X \cup Y \cup V(R_1) \cup \cdots \cup V(R_c)|}{i(G - (W \cup X \cup Y \cup V(R_1) \cup \cdots \cup V(R_c)))} \\
&= \frac{n + |X| + |Y| + \sum_{i=1}^{c} |V(R_i)|}{a + b + \sum_{i=1}^{c} |V(R_i)|} \\
&= \frac{n + |X| + b + \sum_{i=1}^{c} |V(R_i)|}{a + b + \sum_{i=1}^{c} |V(R_i)|},
\end{aligned}
$$

which implies

$$
(n + 3\lambda + 5)a + (n + \lambda + 2)b + (n + \lambda + 2) \sum_{i=1}^{c} |V(R_i)| - (2\lambda + 3)|X| - (2\lambda + 3)n < 0. \qquad (3.6)
$$

By means of (3.1), (3.3), (3.6), $\varepsilon_2(X) \leq 2$ and $|V(R_i)| \geq 3$, we derive

$$
\begin{aligned}
0 &> (n + 3\lambda + 5)a + (n + \lambda + 2)b + (n + \lambda + 2) \sum_{i=1}^{c} |V(R_i)| - (2\lambda + 3)|X| - (2\lambda + 3)n \\
&\geq (n + \lambda + 2)a + (n + \lambda + 2)b + (n + \lambda + 2)c - (2\lambda + 3)|X| - (2\lambda + 3)n \\
&= (n + \lambda + 2)(a + b + c) - (2\lambda + 3)|X| - (2\lambda + 3)n \\
&= (n + \lambda + 2)sun(G' - X) - (2\lambda + 3)|X| - (2\lambda + 3)n \\
&\geq (n + \lambda + 2)(2|X| - \varepsilon_2(X) + 1) - (2\lambda + 3)|X| - (2\lambda + 3)n \\
&\geq (n + \lambda + 2)(2|X| - 1) - (2\lambda + 3)|X| - (2\lambda + 3)n \\
&= (2n + 1)|X| - (\lambda + 2)(2n + 1),
\end{aligned}
$$

which implies that $|X| < \lambda + 2$, which contradicts that $|X| \geq \lambda + 2$. This completes the proof of Theorem 1.6. □

Remark 3.1. Next, we claim that the condition $I(G) > \frac{n+3\lambda+5}{2\lambda+3}$ in Theorem 1.6 is sharp. We construct a graph $G = K_{n+\lambda+2} \vee ((2\lambda + 3)K_2)$, where n and λ are two nonnegative integers, $\vee$ means "join". Obviously, G is $(n + \lambda + 2)$-connected and $I(G) = \frac{n+3\lambda+5}{2\lambda+3}$. For $W \subseteq V(K_{n+\lambda+2})$ with $|W| = n$, let $G' = G - W = K_{\lambda+2} \vee ((2\lambda + 3)K_2)$. Select $X = V(K_{\lambda+2})$ in G'. Then $\varepsilon_2(X) = 2$, and we derive

$$
sun(G' - X) = 2\lambda + 3 > 2\lambda + 2 = 2(\lambda + 2) - 2 = 2|X| - \varepsilon_2(X).
$$

According to Theorem 1.4, G' is not a $P_{\geq 3}$-factor covered graph, and so G is not a $(p_{\geq 3}, n)$-factor-critical covered graph.

Remark 3.2. In what follows, we claim that the condition $(n + \lambda + 2)$-connectivity in Theorem 1.6 is best possible. We construct a graph $G = K_{n+\lambda+1} \vee ((2\lambda + 1)K_2)$, where $n \geq 0$ and $\lambda \geq 1$ are two integers, $\vee$ means "join". Clearly, $I(G) = \frac{n+3\lambda+2}{2\lambda+1} > \frac{n+3\lambda+5}{2\lambda+3}$ and G is $(n+\lambda+1)$-connected. Let $G' = G - W = K_{\lambda+1} \vee ((2\lambda+1)K_2)$, where $W \subseteq V(K_{n+\lambda+1})$ with $|W| = n$. Choose $X = V(K_{\lambda+1})$ in G'. Then $\varepsilon_2(X) = 2$, and we have

$$sun(G' - X) = 2\lambda + 1 > 2\lambda = 2(\lambda + 1) - 2 = 2|X| - \varepsilon_2(X).$$

In light of Theorem 1.4, G' is not a $P_{\geq 3}$-factor covered graph, and so G is not a $(p_{\geq 3}, n)$-factor-critical covered graph.

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References

[1] R. Anstee, Y. Nam, More sufficient conditions for a graph to have factors, Discrete Mathematics 184(1998)15–24.

[2] Y. Egawa, M. Kano, M. Yokota, Existence of all generalized fractional (g, f)-factors of graphs, Discrete Applied Mathematics 283(2020)265–271.

[3] W. Gao, W. Wang, Tight binding number bound for $P_{\geq 3}$-factor uniform graphs, Information Processing Letters 172(2021)106162.

[4] W. Gao, W. Wang, Y. Chen, Tight bounds for the existence of path factors in network vulnerability parameter settings, International Journal of Intelligent Systems 36(2021)1133–1158.

[5] A. Kaneko, A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two, Journal of Combinatorial Theory, Series B 88(2003)195–218.

[6] M. Kano, C. Lee, K. Suzuki, Path and cycle factors of cubic bipartite graphs, Discussiones Mathematicae Graph Theory 28(3)(2008)551–556.

[7] M. Kano, H. Lu, Q. Yu, Component factors with large components in graphs, Applied Mathematics Letters 23(2010)385–389.

[8] K. Kawarabayashi, H. Matsuda, Y. Oda, K. Ota, Path factors in cubic graphs, Journal of Graph Theory 39(2002)188–193.

[9] M. Las Vergnas, An extension of Tutte's 1-factor theorem, Discrete Mathematics 23(1978)241–255.

[10] S. Wang, W. Zhang, On k-orthogonal factorizations in networks, RAIRO-Operations Research 55(2)(2021)969–977.

[11] S. Wang, W. Zhang, Remarks on fractional ID-$[a, b]$-factor-critical covered network graphs, Proceedings of the Romanian Academy, Series A: Mathematics, Physics, Technical Sciences, Information Science 22(3)(2021)209–216.

[12] S. Wang, W. Zhang, Research on fractional critical covered graphs, Problems of Information Transmission 56(2020)270–277.

[13] J. Yang, Y. Ma, G. Liu, Fractional (g, f)-factors in graphs, Appl. Math. J. Chinese Univ. Ser. A 16(2001)385–390.

[14] Y. Yuan, R. Hao, A neighborhood union condition for fractional ID-$[a, b]$-factor-critical graphs, Acta Mathematicae Applicatae Sinica-English Serie 34(4)(2018)775–781.

[15] H. Zhang, S. Zhou, Characterizations for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs, Discrete Mathematics 309(2009)2067–2076.

[16] S. Zhou, A neighborhood union condition for fractional (a, b, k)-critical covered graphs, Discrete Applied Mathematics, DOI: 10.1016/j.dam.2021.05.022

[17] S. Zhou, A result on fractional (a, b, k)-critical covered graphs, Acta Mathematicae Applicatae Sinica-English Series 37(4)(2021)657–664.

[18] S. Zhou, Binding numbers and restricted fractional (g, f)-factors in graphs, Discrete Applied Mathematics 305(2021)350–356.

[19] S. Zhou, Path factors and neighborhoods of independent sets in graphs, Acta Mathematicae Applicatae Sinica-English Series, DOI: 10.1007/s10255-022-1096-2

[20] S. Zhou, Q. Bian, Q. Pan, Path factors in subgraphs, Discrete Applied Mathematics, DOI: 10.1016/j.dam.2021.04.012

[21] S. Zhou, H. Liu, Y. Xu, A note on fractional ID-$[a, b]$-factor-critical covered graphs, Discrete Applied Mathematics, DOI: 10.1016/j.dam.2021.03.004

[22] S. Zhou, Z. Sun, Q. Bian, Isolated toughness and path-factor uniform graphs (II), Indian Journal of Pure and Applied Mathematics, DOI: 10.1007/s13226-022-00286-x

[23] S. Zhou, Z. Sun, H. Liu, On $P_{\geq 3}$-factor deleted graphs, Acta Mathematicae Applicatae Sinica-English Series 38(1)(2022)178–186.

[24] S. Zhou, J. Wu, Q. Bian, On path-factor critical deleted (or covered) graphs, Aequationes Mathematicae 96(4)(2022)795–802.

[25] S. Zhou, J. Wu, Y. Xu, Toughness, isolated toughness and path factors in graphs, Bulletin of the Australian Mathematical Society, DOI: 10.1017/S0004972721000952

[26] S. Zhou, T. Zhang, Z. Xu, Subgraphs with orthogonal factorizations in graphs, Discrete Applied Mathematics 286(2020)29–34.