Analysis of a discrete-time Markov process with a bounded continuous state space by the Fredholm integral equation of the second kind

Koichi Nakade* and Rubayet Karim**
*Nagoya Institute of Technology, Nagoya, Japan
**Jashore University of Science and Technology, Jashore, Bangladesh

Abstract

A discrete-time Markov process with a bounded continuous state space is considered. We show that the equilibrium equations on steady-state probability and densities form Fredholm integral equations of the second kind. Then, under a sufficient condition that the transition densities from one state to another state inside the boundaries of the state space can be expressed in the same separate forms, the steady-state probability and density functions can be obtained explicitly. We use it to demonstrate an economic production quantity model with stochastic production time, derive the expressions of the steady-state probabilities and densities, and find the optimal maximum stock level. A sensitivity analysis of the optimal stock level is performed using production time and cost parameters. The optimal stock level decreases with respect to the holding cost and the production cost, whereas it increases with respect to the lost sale cost and the arrival rate.

Keywords: Markov process, Continuous state, Fredholm integral equation, steady state probability

1. Introduction

A Markov process can be applied to simulate a random system that changes system conditions based on a transition rule that is only dependent on the present state. The most significant property of the Markov process is the conditional probability distribution of the process's future states, which depends only on the current state (Komorowski and Szarek [12], Werner [18]). A discrete-time Markov process model with a continuous state space is applied to many systems, such as a liquid store model, electricity model, autoregressive (AR) model, and so on. It is also useful as the approximate model of the Markov process with a discrete space. The steady-state distributions of the Markov model

This provisional PDF is the accepted version. The article should be cited as: RAIRO: RO, doi: 10.1051/ro/2022099

satisfy the equilibrium equations, but in almost cases, those cannot be represented in an explicit form. In this paper, Fredholm integral equations are applied for analysis of a general discrete-time Markov process with a bounded continuous state space, and we give a sufficient condition for obtaining the steady state probability and density functions.

1.1 Background

In the late nineteenth century, Fredholm and Volterra largely establish the theory of integral equations. Their works have a significant impact on the study of integral equations in the twentieth century. Numerous mathematical models in engineering and science, such as the anomalous diffusion problem, biological population ecological model, and population prediction model, can be described by an integral equation model (Guo [9]). The Fredholm integration equation is found in the theory of signal processing, linear forward modelling, and inverse problems. Fluid mechanics issues involving hydrodynamic interactions near finite-sized elastic interfaces also use Fredholm integral equations (Daddi-Moussa-Ider *et al.* [2], Daddi-Moussa-Ider [3]). A special use of the Fredholm equation is the creation of photorealistic images in computer graphics, in which the Fredholm equation is used to represent light transport from virtual light sources to the image plane. Due to the fact that a vast class of initial and boundary value problems can be transformed into Volterra or Fredholm integral equations, many scientific domains use Fredholm integral equations, including engineering, applied mathematics, and mathematical physics (Gutiérrez, [10]). Other literature regarding the Fredholm integral equations focuses on efficient numerical solution techniques for the Fredholm integral equations (ZhiMin *et al.* [19], Doucet *et al.* [4], Tian [17], Mohammad [14]).

Recently, the Fredholm integral equations have been applied to a specific stochastic process. In particular, Lindemann and Thümmler [13] provide a generic state-space Markov chain (GSSMC) approach for the transient analysis of deterministic and stochastic Petri nets with concurrently enabled deterministic transitions as an application of the Fredholm equation to the study of the stochastic process. The general state-space Markov chain approach is built on a numerical iterative solution of a system of Fredholm integral equations. Fuh *et al.* [8] use the Fredholm integral equation to compute the R´enyi divergence of two-state Markov switching models. Ramsden and Papaioannou [15] derive a Fredholm integral equation of the second kind for the ultimate ruin probability and achieve a clear expression in terms of ruin quantities for the Cramér–Lundberg risk model. Later, they extend the capital injection delayed risk model such that the delay of the capital injections depends explicitly on the amount of the deficit. Dibu *et al.* [5] introduce a Markov Arrival Process risk model that permits capital injections to be received promptly or with an arbitrary delay, depending on the amount of shortage experienced by the firm. For this model, they originate a system of Fredholm integral equations of the second kind for the Gerber-Shiu function and derive a straightforward formulation in matrix form in terms of the Gerber-Shiu function of the Markov Arrival Process risk model.

1.2 Motivation, Research Gap and Objective

Generally, the steady-state distributions of the Markov model cannot be represented in an explicit form. From literature, for the specified model, the analysis is made theoretically by using Fredholm integral equations. To the best of our knowledge, there is currently no mathematical approach that uses the Fredholm integral equation of the second kind to study the steady state of a general discrete-time Markov process with a bounded continuous state space. Recently, Karim and Nakade [11] show such an application in a special case of an economic production quantity (EPQ) model. They derive the expression of the steady-state distribution by using Fredholm integral equations. Although the model they analyse is a restricted one, their analysis suggests that the general discrete-time Markov process with continuous states can also be analysed by the Fredholm integral equation.

Thus motivated, the application of the Fredholm integral equation of the second kind for the steady-state analysis of a general discrete-time Markov process with a bounded continuous state space is studied in this paper. Then, as an example application, we analyse a fundamental EPQ model. We show that the Fredholm integral equation of the second kind can be used to express the equilibrium equations on steady-state probability densities. The functions of the Fredholm integral equations can be solved using the degenerate kernel method when some function satisfies separable properties (Raisinghania, [16]).

The contribution of this paper is as follows:

- For a general discrete-time Markov process with a bounded continuous state space, we derive sufficient conditions on separable properties of transition density functions, under which expressions of the steady-state probabilities and densities can be derived explicitly.
- Then, as an application, we discuss a basic EPQ model, apply the derived analytical method, and derive the optimal size of the maximum inventory level.

The organization of this paper is as follows: In Section 2, the Fredholm integration equation of the second type is described. In Section 3, it is shown that this equation can be applied to the analysis of the Markov process with a bounded state space under some separable conditions on the transition densities and probabilities. In Section 4, the analysis in Section 3 is applied to the basic EPQ model, and the optimal maximum inventory level is derived. In Section 5, we give the conclusion.

2. Fredholm integral equation of the second kind

The Fredholm integral equation of the second kind is found in the theory of signal processing, linear forward modelling, and inverse problems. The equation on the function $f(r)$ is given as follows.

$$f(r) = u(r) + \int_a^b K(r, r')f(r')dr', \quad a < r < b.$$

In the following, we set $a = 0$ and $b = R$. That is,

$$f(r) = u(r) + \int_0^R K(r, r')f(r')dr', \quad 0 < r < R. \tag{1}$$

The degenerate kernel method (Raisinghania [16]) can be applied when $K(r, r')$ satisfies the following separate form:

$$K(r, r') = \sum_{j=0}^{m-1} w_j(r)g_j(r'), \tag{2}$$

where $w_j(r)$ and $g_j(r')$ are functions of only r and r', respectively. Then, by inserting (2) into the function inside the integral in (1), we have

$$f(r) = u(r) + \sum_{j=0}^{m-1} w_j(r)C_j, \tag{3}$$

where $C_j = \int_0^R g_j(r')f(r')dr' \ (j = 0,1, \dots, m-1)$.

Set

$$\gamma_{ij} = \int_0^R g_i(r)w_j(r)dr \ (i, j = 0,1, \dots, m-1), \tag{4}$$

and

$$\beta_i = \int_0^R g_i(r)u(r)dr \ (i = 0,1, \dots, m-1). \tag{5}$$

By multiplying $g_i(r)$ on both sides of (3) and integrating it from 0 to R, we have the following equations.

$$(1-\gamma_{ii})C_i - \sum_{j\neq i}\gamma_{ij}C_j = \beta_i, \qquad i = 0,1,\dots,m-1. \tag{6}$$

Thus, if γ_{ij} and β_i can be computed for all i and j, we can derive C_i by solving (6). Here, we consider the case $m = 2$. Then

$$K(r,r') = w_0(r)g_0(r') + w_1(r)g_1(r'),$$

$$f(r) = u(r) + w_0(r)C_0 + w_1(r)C_1, \tag{7}$$

$$C_0 = \int_0^R g_0(r')f(r')dr', \qquad C_1 = \int_0^R g_1(r')f(r')dr',$$

$$(1-\gamma_{00})C_0 - \gamma_{01}C_1 = \beta_0, \quad -\gamma_{10}C_0 + (1-\gamma_{11})C_1 = \beta_1, \tag{8}$$

where

$$\gamma_{00} = \int_0^R g_0(r)w_0(r)dr, \quad \gamma_{01} = \int_0^R g_0(r)w_1(r)dr, \quad \gamma_{10} = \int_0^R g_1(r)w_0(r)dr,$$

$$\gamma_{11} = \int_0^R g_1(r)w_1(r)dr, \quad \beta_0 = \int_0^R g_0(r)u(r)dr, \quad \beta_1 = \int_0^R g_1(r)u(r)dr.$$

When $(1-\gamma_{00})(1-\gamma_{11}) - \gamma_{01}\gamma_{10} \neq 0$, a unique non-zero solution of the system of Equations (8) exists and is given by

$$C_0 = \frac{(1-\gamma_{11})\beta_0 + \gamma_{01}\beta_1}{(1-\gamma_{00})(1-\gamma_{11}) - \gamma_{01}\gamma_{10}} \text{ and } C_1 = \frac{\gamma_{10}\beta_0 + (1-\gamma_{00})\beta_1}{(1-\gamma_{00})(1-\gamma_{11}) - \gamma_{01}\gamma_{10}}. \tag{9}$$

Thus, we obtain $f(r)$ by (7).

3. A discrete time Markov process with a bounded continuous state space

We consider a discrete-time Markov process with bounded continuous state space. Without loss of generality, the state space is set as $S = [0,R]$, the transition intensity (stochastic kernel) from $r' \in S$ to $r \in (0,R)$ is $q_{r',r}$, and the transition probabilities from $r \in S$ to 0 and R are q_{r0} and q_{rR}, respectively. Let steady-state mass probabilities in states 0 and R be π_0 and π_R, respectively, and let the steady-state probability density in state $r \in (0,R)$ be denoted by $f(r)$.

The Markov process assumes that the state space forms one recurrent class. This implies that $q_{00} < 1$ and $q_{RR} < 1$. Then, we have the following equilibrium equations (see Feller [7], for example).

$$\pi_0 = \pi_0 q_{00} + \pi_R q_{R0} + \int_0^R f(r') q_{r'0}\, dr', \tag{10}$$

$$f(r) = \pi_0 q_{0r} + \pi_R q_{Rr} + \int_0^R f(r') q_{r'r} dr', \quad 0 < r < R, \tag{11}$$

$$\pi_R = \pi_0 q_{0R} + \pi_R q_{RR} + \int_0^R f(r') q_{r'R} dr'. \tag{12}$$

We also have the total probability of 1 and thus

$$\pi_R + \pi_0 + \int_0^R f(r) dr = 1. \tag{13}$$

Note that (10) can be obtained from (11) and (12) because $q_{r'0} + q_{r'R} + \int_0^R q_{r'r} dr = 1$ for $r' \in S$. From Equations (12) and (13),

$$\pi_R = \left(1 - \pi_R - \int_0^R f(r) dr\right) q_{0R} + \pi_R q_{RR} + \int_0^R f(r') q_{r'R} dr',$$

and thus,

$$\pi_R = \frac{1}{1 - q_{RR} + q_{0R}} \left\{ q_{0R} - q_{0R} \int_0^R f(r') dr' + \int_0^R f(r') q_{r'R} dr' \right\}. \tag{14}$$

Inserting Equations (13) and (14) into (11), we have

$$f(r) = q_{0r}[1 - \frac{1}{1 - q_{RR} + q_{0R}} \left\{ q_{0R} - q_{0R} \int_0^R f(r') dr' + \int_0^R f(r') q_{r'R} dr' \right\}$$

$$- \int_0^R f(r') dr'] + \frac{q_{Rr}}{1 - q_{RR} + q_{0R}} \{ q_{0R} - q_{0R} \int_0^R f(r') dr'$$

$$+ \int_0^R f(r') q_{r'R} dr' \} + \int_0^R f(r') q_{r'r} dr'$$

$$= \frac{q_{Rr} - q_{0r}}{1 - q_{RR} + q_{0R}} q_{0R} + q_{0r}$$

$$+ \int_0^R f(r') \left[\frac{1}{1 - q_{RR} + q_{0R}} \{ -q_{Rr} q_{0R} + q_{Rr} q_{r'R} + q_{0r} q_{0R} - q_{0r} q_{r'R} \} - q_{0r} \right.$$

$$\left. + q_{r'r} \right] dr'$$

$$= \frac{q_{Rr} q_{0R} + q_{0r}(1 - q_{RR})}{1 - q_{RR} + q_{0R}}$$

$$+ \int_0^R f(r') \left[\frac{-q_{Rr} q_{0R} - q_{0r}(1 - q_{RR})}{1 - q_{RR} + q_{0R}} + \frac{q_{Rr} - q_{0r}}{1 - q_{RR} + q_{0R}} q_{r'R} + q_{r'r} \right] dr',$$

$$0 < r < R. \qquad (15)$$

We now give a condition under which the steady-state density can be derived.

Proposition 1. For a discrete-time Markov process with bounded continuous real state space $[0, R]$, if the transition probability density $q_{r'r}$ can be represented as $q_{r'r} = \sum_{i=2}^{n} w_i(r) g_i(r')$, for all $r, r' \in (0, R)$, the steady-state density function $f(r)$ can be derived explicitly by the degenerate kernel method. In addition, if $\int_0^R f(r) dr$, $\int_0^R f(r) q_{rR} dr$ and $\int_0^R f(r) q_{r0} dr$ can be expressed explicitly, the steady-state probabilities π_0 and π_R can also be derived explicitly.

Proof: Comparing (15) with equations in the previous section, by setting

$$u(r) = \frac{q_{Rr} q_{0R} + q_{0r}(1 - q_{RR})}{1 - q_{RR} + q_{0R}}, \qquad (16)$$

$$K(r, r') = q_{r'r} + \frac{-q_{Rr} q_{0R} - q_{0r}(1 - q_{RR})}{1 - q_{RR} + q_{0R}} + \frac{q_{Rr} - q_{0r}}{1 - q_{RR} + q_{0R}} q_{r'R}, \qquad (17)$$

we have

$$f(r) = u(r) + \int_0^R K(r, r') f(r') dr', \qquad 0 < r < R,$$

which is the same as the Fredholm equation of the second kind. Thus, if $K(r, r')$ is represented as $K(r, r') = \sum_{i=0}^{m-1} w_i(r) g_i(r')$, we can apply the degenerate kernel method shown in Section 2 and derive the probability density $f(r)$. If $q_{r'r}$ is represented as $q_{r'r} = \sum_{i=2}^{n} w_i(r) g_i(r')$, for all $r, r' \in (0, R)$, the term in the brace of Equation (15) is formed as $\sum_{i=0}^{n} w_i(r) g_i(r')$, where

$$w_0(r) = \frac{-q_{Rr} q_{0R} - q_{0r}(1 - q_{RR})}{1 - q_{RR} + q_{0R}}, \qquad g_0(r') = 1,$$

$$w_1(r) = \frac{q_{Rr} - q_{0r}}{1 - q_{RR} + q_{0R}}, \qquad g_1(r') = q_{r'R}.$$

If we can derive expressions of integrals $\int_0^R f(r)dr$, $\int_0^R f(r) q_{rR} dr$ and $\int_0^R f(r) q_{r0} dr$ explicitly, the steady-state probabilities π_R and π_0 can also be represented explicitly by (13) and (14). Q.E.D.

Note that when $n = 2$ and $q_{r'r}$ has a special form, such as $g_0(r') = g_2(r')$, the term can be represented as $w_0(r) g_0(r') + w_1(r) g_1(r')$.

Here, we discuss a sufficient condition under which $q_{r'r} = \sum_{i=2}^{n} w_i(r) g_i(r')$, for all $r, r' \in (0, R)$. We assume the following.

Assumption A For some $n \geq 2$, there exist real numbers a_i, b_i and a nonnegative integer s_i for each $i \in 0,1,\ldots,n-1$ that satisfy

$$q_{r'r} = \sum_{i=0}^{n-1} a_i (r - r')^{s_i} e^{-b_i(r-r')} \quad for\ 0 < r < R, 0 \leq r' \leq R,$$

and there exists a nonnegative integer m, real numbers a_i', b_i' and a nonnegative integer s_i' for each $i \in 0,1,\ldots,m-1$ that satisfy

$$q_{r'0} = \sum_{i=0}^{m-1} a_i' (r')^{s_i'} e^{b_i' r'} \quad for\ 0 \leq r' \leq R. \tag{18}$$

For example, Assumption A is satisfied when the transition density $q_{r'r}$ from r' to r for all $r, r' \in (0, R)$ only depends on the difference between r and r' and it is given as an Erlang-type distribution with parameter (n, λ) as

$$q_{r'r} = \frac{\lambda^n (r - r' + a)^{n-1}}{(n-1)!} e^{-\lambda(r - r' + a)}$$

$$= \sum_{i=0}^{n-1} \frac{\lambda^n a^{n-1-i}}{k!\,(n-1-i)!} e^{-\lambda a} (r - r')^i e^{-\lambda(r-r')} \quad \text{for } 0 < r < R, 0 \leq r' \leq R,$$

and

$$q_{r'0} = 1 - \sum_{j=0}^{n-1} \frac{\lambda^j(-r'+a)^j}{j!} e^{-\lambda(-r'+a)} = 1 - e^{-\lambda(-r'+a)} \sum_{j=0}^{n-1} \lambda^j \sum_{l=0}^{j} \frac{(-r')^l a^{j-l}}{l!\,(j-l)!}$$

$$\text{for } \ 0 \le r' \le R$$

where $a \ge R$. Assumption A is also satisfied when $q_{r'r}$ is given as a hyperexponential distribution as

$$q_{r'r} = \sum_{i=0}^{n-1} p_i \lambda_i e^{-\lambda_i(r-r'+a)}, \quad \text{where} \sum_{i=0}^{n-1} p_i = 1 \text{ and } a \ge R.$$

Corollary 1. When assumption A is satisfied, we can derive expressions $f(r)$, π_0, and π_R explicitly.

Proof. Under Assumption A, $q_{r'r}$ can be represented as follows:

$$q_{r'r} = \sum_{i=0}^{n-1} a_i (r-r')^{s_i} e^{-b_i(r-r')} = \sum_{i=0}^{n-1} \sum_{j=0}^{s_i} a_i \binom{s_i}{j} r^j e^{-b_i r} (-r')^{s_i-j} e^{b_i r'}$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{s_i} w_{ij}^1(r) g_{ij}^1(r')$$

where

$$w_{ij}^1(r) = a_i \binom{s_i}{j} r^j e^{-b_i r} \qquad j = 0,1,..,s_i - 1, i = 0,1,\dots,n-1, \tag{19}$$

$$g_{ij}^1(r') = (-r')^{s_i-j} e^{b_i r'} \quad j = 0,1,..,s_i - 1, i = 0,1,\dots,n-1. \tag{20}$$

Thus, from Proposition 1, $f(r)$ for $r \in (0, R)$ can be derived explicitly. In fact, when $w_{ij}^1(r)$ and $g_{ij}^1(r')$ are given by (19) and (20), respectively, in the same way as (4) and (5),

$$\gamma_{(ij),(kl)} = \int_0^R g_{ij}^1(r) w_{kl}^1(r) dr = \int_0^R (-r)^{s_i-j} e^{b_i r} a_k \binom{s_k}{l} r^l e^{-b_k r} dr$$

$$= a_k \binom{s_k}{l} (-1)^{s_i-j} \int_0^R r^{s_i-j+l} e^{(b_i-b_k)r} dr$$

and since $u(r) = \frac{q_{0R}}{1-q_{RR}+q_{0R}} q_{Rr} + \frac{1-q_{RR}}{1-q_{RR}+q_{0R}} q_{0r}$, we have

$$u(r) = c_R \sum_{i=0}^{n-1} \sum_{j=0}^{s_i} w_{ij}^1(r) g_{ij}^1(R) + c_0 \sum_{i=0}^{n-1} \sum_{j=0}^{s_i} w_{ij}^1(r) g_{ij}^1(0)$$

where $c_R = \frac{q_{0R}}{1-q_{RR}+q_{0R}}$ and $c_0 = \frac{1-q_{RR}}{1-q_{RR}+q_{0R}}$, and thus

$$\beta_{ij} = \int_0^R g_{ij}^1(r)u(r)dr = c_R \int_0^R (-r)^{s_i-j} e^{b_i r} \sum_{k=0}^{n-1} \sum_{l=0}^{s_k} a_k \binom{s_k}{l} r^l e^{-b_k r} (-R)^{s_k-l} e^{b_k R} dr$$

$$+c_0 \int_0^R (-r)^{s_i-j} e^{b_i r} \sum_{k=0}^{n-1} a_k r^{s_k} e^{-b_k r} dr$$

$$= c_R \sum_{k=0}^{n-1} \sum_{l=0}^{s_k} a_k \binom{s_k}{l} (-R)^{s_k-l} e^{b_k R} (-1)^{s_i-j} \int_0^R r^{s_i-j+l} e^{(b_i-b_k)r} dr$$

$$+c_0 \sum_{k=0}^{n-1} a_k (-1)^{s_i-j} \int_0^R r^{s_i-j+s_k} e^{(b_i-b_k)r} dr$$

Since $\int_0^R r^m e^{ar} dr$ is easily computable for integer $m \geq 0$ and real number a, $\gamma_{(ij),(kl)}$ and $\beta_{(ij)}$ can be computed. Thus, in the same way as Section 2, we can compute values of the set $\{C_{ij}; i,j = 0,1,\dots,m-1\}$ which satisfy

$$\left(1-\gamma_{(ij)(ij)}\right)C_{(ij)} - \sum_{(kl)\neq(ij)} \gamma_{(ij)(kl)} C_{(ij)} = \beta_{(ij)}, \qquad i,j = 0,1,\dots,m-1.$$

and then we have

$$f(r) = u(r) + \sum_{i=0}^{n-1} \sum_{j=0}^{s_i} w_{ij}^1(r) C_{ij} \qquad\qquad 0 < r < R.$$

From the forms of $u(r)$ and $w_{ij}^1(r)$, $\int_0^R f(r)dr$ can be computed, and since q_{r0} satisfies (18), $\int_0^R f(r) q_{r0} dr$ can be obtained in the same way as above. Thus, π_0 can be computed, and π_R is also derived by (13). Q.E.D.

4. The application of the analytical method to an EPQ model

4.1 Analysis of an EPQ model

The modelling of production-inventory systems is one of the most important applications of the Markov process. Revenue is earned when product supply meets customer demand. Inventory control is primarily related to the matching of supply and demand. Some of the earliest research papers on inventory system modelling with Markovian models are from the 1950s (see, for example, Arrow et al. [1], Fabens [6]). Markovian models have since gained a good deal of popularity in inventory control.

In this section, as an application of result established in the previous section, we consider the following EPQ model and derive the optimal upper limit of the items in inventory.

One unit period corresponds to one day. In each period, the amount of demand of items is D, and it is fixed. The manufacturer produces items. The process may fail, and X is the possible number of items produced by the epoch when the failure occurs. The upper bound of items in inventory is R. Here, we assume that $D \geq R$. At the beginning of each day, the production restarts even if the system had failed in the previous day by repairing the process at the end of the previous day (see Fig. 1).

When the number of items in inventory is r' at the beginning of each period, if the number of items produced in this period reaches $R + D - r'$, the system stops production, and after the demand of this period is met, the number of items in inventory becomes R. When the number of possible produced products is X, the number of items in inventory, denoted by r, is given by

$$r = \max(0, r' + X - D).$$

Thus,

(i) when $X \geq R + D - r'$, $r = R$,

(ii) when $D - r' < X < R + D - r'$, $r = r' + X - D \in (0, R)$, and

(iii) when $X < D - r'$, $r = 0$.

In the last case, the excess demand is lost. Note that $r' \leq R < D$ implies that the last case is possible. Then, we have for each $r' \in [0, R]$

$$q_{r'0} = F_X(D - r'), \quad q_{r'r} = f_X(r - r' + D) \text{ for } 0 < r < R,$$

$$q_{r'R} = 1 - F_X(R + D - r').$$

Thus, when X follows an Erlang-type distribution, the sufficient condition of Corollary 1 in Section 3 is satisfied. Thus, we can derive the steady-state probability density $f(r)$ and steady-state mass probability functions π_0 and π_R.

In the following, we study the case where X follows an exponential distribution with parameter λ. This is satisfied when the failure rate is constant and the failure is not unpredictable. Then, $f_X(x) = \lambda\, e^{-\lambda\, x}$ is a density function of X and $F_X(x) = 1 - e^{-\lambda\, x}$ is a distribution function of X. Thus, we have for each $r' \in [0, R]$

$$q_{r'r} = \lambda e^{-\lambda\,(r+D)} e^{\lambda r'}, q_{r'0} = 1 - e^{-\lambda\,(D-r')}, q_{r'R} = e^{-\lambda\,(R+D-r')}.$$

By (16) and (17),

$$u(r) = \frac{\lambda e^{-\lambda\,(r+D)} e^{\lambda R} \cdot e^{-\lambda\,(R+D)} + \lambda e^{-\lambda\,(r+D)}(1 - e^{-\lambda\, D})}{1 - e^{-\lambda\, D} + e^{-\lambda\,(R+D)}} = \frac{\lambda e^{-\lambda\,(r+D)}}{1 - e^{-\lambda\, D} + e^{-\lambda\,(R+D)}},$$

$$K(r,r') = \lambda e^{-\lambda\,(r+D)} e^{\lambda r'} + \frac{-\lambda e^{-\lambda\,(r+D)} e^{\lambda R}\cdot e^{-\lambda\,(R+D)} - \lambda e^{-\lambda\,(r+D)}\left(1-e^{-\lambda\,D}\right)}{1-e^{-\lambda\,D}+e^{-\lambda\,(R+D)}}$$

$$+\frac{\lambda e^{-\lambda\,(r+D)}\left(e^{\lambda R}-1\right)\cdot e^{-\lambda\,(R+D-r')}}{1-e^{-\lambda\,D}+e^{-\lambda\,(R+D)}}$$

$$= \lambda e^{-\lambda\,(r+D)} e^{\lambda r'} + \frac{-\lambda e^{-\lambda\,(r+D)} + \lambda e^{-\lambda\,(r-r'+2D)} - \lambda e^{-\lambda\,(r-r'+R+2D)}}{1-e^{-\lambda\,D}+e^{-\lambda\,(R+D)}}$$

$$= \frac{\lambda e^{-\lambda\,(r+D)}}{1-e^{-\lambda\,D}+e^{-\lambda\,(R+D)}}\left(-1+e^{\lambda\,r'}\right) = w_0(r)g_0(r') + w_1(r)g_1(r'),$$

where

$$w_0(r) = \frac{-\lambda e^{-\lambda\,(r+D)}}{1-e^{-\lambda\,D}+e^{-\lambda\,(R+D)}},\ g_0(r') = 1,\ w_1(r) = -w_0(r),\ g_1(r') = e^{\lambda\,r'}.$$

Let $Y = 1-e^{-\lambda\,D}+e^{-\lambda\,(R+D)}$, and then

$$\gamma_{00} = \frac{-e^{-\lambda D}\left(1-e^{-\lambda R}\right)}{Y},\qquad \gamma_{01} = \frac{e^{-\lambda D}\left(1-e^{-\lambda R}\right)}{Y},\qquad \gamma_{10} = \frac{-\lambda R e^{-\lambda D}}{Y},$$

$$\gamma_{11} = \frac{\lambda R e^{-\lambda D}}{Y},\qquad \beta_0 = \frac{e^{-\lambda D}\left(1-e^{-\lambda R}\right)}{Y},\qquad \beta_1 = \frac{\lambda R e^{-\lambda D}}{Y}.$$

Since $xe^{-x} < 1$ for $x > 0$,, we have $\lambda R e^{-\lambda D} < 1$ because it has been assumed that $R \leq D$. From (9)

$$C_0 = \frac{e^{-\lambda D}\left(1-e^{-\lambda R}\right)}{1-\lambda R e^{-\lambda D}},\qquad \text{and}\quad C_1 = \frac{\lambda R e^{-\lambda D}}{1-\lambda R e^{-\lambda D}}.$$

Thus, by (3), we have

$$f(r) = \frac{\lambda e^{-\lambda\,(r+D)}}{1-e^{-\lambda\,D}+e^{-\lambda\,(R+D)}} + \frac{-\lambda e^{-\lambda\,(r+D)}}{1-e^{-\lambda\,D}+e^{-\lambda\,(R+D)}}\frac{e^{-\lambda D}\left(1-e^{-\lambda R}\right)}{1-\lambda R e^{-\lambda D}}$$

$$+\frac{\lambda e^{-\lambda\,(r+D)}}{1-e^{-\lambda\,D}+e^{-\lambda\,(R+D)}}\frac{\lambda R e^{-\lambda D}}{1-\lambda R e^{-\lambda D}} = \frac{\lambda e^{-\lambda\,(r+D)}}{1-\lambda R e^{-\lambda D}},\quad \text{for } r \in (0,R).$$

Since $\int_0^R f(r)dr = \frac{e^{-\lambda D}(1-e^{-\lambda R})}{1-\lambda R e^{-\lambda D}}$, we have

$$\pi_R = \frac{e^{-\lambda\,(R+D)} + \frac{1}{1-\lambda R e^{-\lambda D}}\left(-e^{-\lambda\,(R+2D)} + e^{-\lambda\,(2R+2D)} + \lambda R e^{-\lambda\,(R+2D)}\right)}{1-e^{-\lambda\,D}+e^{-\lambda\,(R+D)}}$$

$$= \frac{e^{-\lambda\,(R+D)}}{1-\lambda R e^{-\lambda D}},$$

$$\pi_0 = \frac{1-e^{-\lambda D}-\lambda R e^{-\lambda D}}{1-\lambda R e^{-\lambda D}}.$$

4.2 An optimization problem

For a given R, we derive the average cost by using the derived limiting distributions of the number of items in inventory. Let the production cost per unit item, the holding cost rate per unit time, and the lost sale cost for each lost demand be $c, h,$ and l, respectively. We also define a function as for $a \geq 0$

$$g(a) = E[\min(X, a)] = \int_0^a x \cdot \lambda e^{-\lambda x} dx + a \cdot e^{-\lambda a} = \frac{1}{\lambda}\left(1 - e^{-\lambda a}\right).$$

Note that $E[\max(0, a - X)] = E[a - \min(a, X)] = a - g(a)$.

By using equations

$$\int_0^a (a - x)\lambda e^{-\lambda x} dx = a - \frac{1}{\lambda}\left(1 - e^{-\lambda a}\right),$$

$$\int_0^a \lambda x e^{-\lambda x} dx + a e^{-\lambda a} = \frac{1}{\lambda}\left(1 - e^{-\lambda a}\right), \text{ for } a \geq 0,$$

we obtain the following average cost per unit time.

(a) Average production cost

$$c\left(\pi_0 E[\min(X, R + D)] + \int_0^R E[\min(X, R + D - r')] f(r') dr' + \pi_R E[\min(X, D)]\right)$$

$$= c\,\frac{1}{1 - \lambda R e^{-\lambda D}}\Bigg((1 - e^{-\lambda D} - \lambda R e^{-\lambda D}) g(R + D) + \int_0^R g(R + D - r')\,\lambda e^{-\lambda\,(r' + D)} dr' + e^{-\lambda\,(R + D)} g(D)\Bigg)$$

$$= c\left(\frac{1}{\lambda(1 - \lambda R e^{-\lambda D})} \cdot \left(1 - \lambda R e^{-\lambda D} - e^{-\lambda(R + D)}\right)\right)$$

(b) Average lost sale cost

$$l\left(\pi_0 E[\max(0, D - X)] + \int_0^R E[\max(0, D - r' - X)] f(r') dr' + \pi_R E[\max(0, D - R - X)]\right)$$

$$= l\,\frac{1}{1 - \lambda R e^{-\lambda D}}\Bigg((1 - e^{-\lambda D} - \lambda R e^{-\lambda D})\left(D - g(D)\right) + \int_0^R \left(D - r' - g(D - r')\right)\lambda e^{-\lambda\,(r' + D)}\, dr' + e^{-\lambda\,(R + D)}(D - R - g(D - R))\Bigg)$$

$$= l \cdot \frac{1}{1 - \lambda R e^{-\lambda D}}\left(D - \frac{1}{\lambda} + R(1 - \lambda D) e^{-\lambda D} + \frac{1}{\lambda} e^{-\lambda(R + D)}\right)$$

(c) Average holding cost

$$h\left(\int_0^R r\,f(r)dr + \pi_R R\right) = h\left(\int_0^R \frac{\lambda r' e^{-\lambda(r'+D)}}{1-\lambda R e^{-\lambda D}}dr' + \frac{R e^{-\lambda(R+D)}}{1-\lambda R e^{-\lambda D}}\right) = h\cdot\frac{e^{-\lambda D}\left(1-e^{-\lambda R}\right)}{\lambda\left(1-\lambda R e^{-\lambda D}\right)}.$$

Thus, the average cost for a given R is given by

$$C(R) = \frac{1}{\lambda(1-\lambda R e^{-\lambda D})}\Big(c\big(1-\lambda R e^{-\lambda D} - e^{-\lambda(R+D)}\big) + h e^{-\lambda D}\big(1-e^{-\lambda R}\big)$$

$$+ l\big(\lambda D - 1 + \lambda R(1-\lambda D)e^{-\lambda D} + e^{-\lambda(R+D)}\big)\Big). \qquad (21)$$

We derive the optimal R. By differentiating both sides of (21),

$$\frac{d}{dR}C(R) = \frac{e^{-\lambda(R+2D)}}{\lambda^2(1-\lambda R e^{-\lambda D})^2}\Big(c\big(-1-\lambda R + e^{\lambda D}\big) + h\big(-1-\lambda R + e^{\lambda R} + e^{\lambda D}\big)$$

$$+ l\big(1+\lambda R - e^{\lambda D}\big)\Big)$$

$$= \frac{e^{-\lambda(R+2D)}}{\lambda^2(1-\lambda R e^{-\lambda D})^2}\Big(\big(l-(h+c)\big)\big(\lambda R + 1 - e^{\lambda D}\big) + h e^{\lambda R}\Big)$$

$$= \frac{e^{-\lambda(R+2D)}\big(l-(h+c)\big)}{\lambda^2(1-\lambda R e^{-\lambda D})^2}\Big(\big(\lambda R + 1 - e^{\lambda D}\big) + \frac{h}{l-(h+c)}e^{\lambda R}\Big).$$

We assume that $l-(h+c) > 0$. This is not restrictive because the lost sale cost is higher than a production cost and unit time holding cost. Otherwise, the system will not produce a product.

Let $v(x) = \big(x + 1 - e^{\lambda D}\big) + \frac{h}{l-(h+c)}e^{x}$. Then, we know that

$$v(0) = 1 - e^{\lambda D} + \frac{h}{l-(h+c)} = \frac{l-c}{l-(h+c)} - e^{\lambda D},\ v'(x) = 1 + \frac{h}{l-(h+c)}e^{x} > 0,$$

$$v(\lambda D) = \lambda D + 1 - \frac{l-(2h+c)}{l-(h+c)}e^{\lambda D}.$$

Thus, if $\frac{l-c}{l-(h+c)} < e^{\lambda D}$ and $\lambda D + 1 > \frac{l-(2h+c)}{l-(h+c)}e^{\lambda D}$, there is an optimal $R^* \in (0, D)$ which is a unique solution of $v(\lambda R) = 0$. If $\frac{l-c}{l-(h+c)} \geq e^{\lambda D}$, $R^* = 0$, and if $\lambda D + 1 \leq \frac{l-(2h+c)}{l-(h+c)}e^{\lambda D}$ then $R^* = D$ in the interval $[0, D]$ of R.

4.3 Sensitivity Analysis

Here, we perform sensitivity analysis on R^*. We note that R^* satisfies $v(\lambda R^*) = 0$. By letting $d = -1 + e^{\lambda D}$ and $B = \frac{h}{l-(h+c)}$, R^* satisfies $Be^{\lambda R^*} = d - \lambda R^*$. Fig. 2 shows that λR^* is the intersection point of $y = Be^x$ and $y = d - x$. Since B increases in h and c while decreasing in l, from Fig. 2, we find that R^*decreases in h and c while increasing in l. In addition, it is observed from Fig. 1 that R^* increases in D.

The optimal R^* can be considered a function of λ. Since $Be^{\lambda R^*} = -1 + e^{\lambda D} - \lambda R^*$, by differentiating with respect to λ, we have

$$Be^{\lambda R^*}\left(R^* + \lambda \frac{dR^*}{d\lambda}\right) + R^* + \lambda \frac{dR^*}{d\lambda} - De^{\lambda D} = 0,$$

which implies

$$\frac{dR^*}{d\lambda} = \frac{De^{\lambda D} - R^*\left(Be^{\lambda R^*} + 1\right)}{\lambda\left(Be^{\lambda R^*} + 1\right)}.$$

Since $De^{\lambda D} - R^*\left(Be^{\lambda R^*} + 1\right) = (D - R^*)e^{\lambda D} + \lambda R^{*2}$, R^* increases in λ when R^* is in $(0, D)$.

The relationship between the optimal bound R^* and the arrival rate λ is depicted in Fig. 3. Here, we set $c = 4$, $l = 10$, $h = 1$ and thus $B = 0.2$, and $D = 5$. In this case, when$\lambda \leq 0.035$, $\frac{l-c}{l-(h+c)} \geq e^{\lambda D}$, and $R^* = 0$ is optimal. When $\lambda \geq 0.165$, we can show that $\lambda D + 1 \leq \frac{l-(2h+c)}{l-(h+c)} e^{\lambda D}$. Thus, for $0.04 \leq \lambda \leq 0.16$,, the optimal curve is shown in Fig. 3. In this interval of λ, the optimal R^* is concave in λ, although we cannot show the concavity theoretically.

It should be highlighted that when $R > D$ is allowed, a lower average cost can be reached. This occurs when the probability of a short production time is high, and it is preferable to store more items to prepare for the subsequent short production times. In this case, however, the analytical method in this paper cannot be applied because the expression of transition intensity $q_{r'r}$ depends on the combination of r’ and r.

5. Conclusion

This paper considers the application of a Fredholm integral equation of the second kind to the analysis of a discrete-time Markov process with a continuous state space having a finite interval. We first show that the equilibrium equations on the steady-state mass and density probability functions are formed as the Fredholm integral equation. Then, under some separable conditions, the transition density from r' to r in the inner state-space forms the product of two functions that are functions of only r' and r, respectively, we can obtain expressions of these probability functions explicitly. As an example, a basic EPQ model is analysed, and its optimal bound is developed. The optimal upper bound is derived, and we analytically study the sensitivity of the optimal bound. The bound decreases with respect to the holding cost and the production cost, whereas it increases with respect to the lost sale cost and the arrival rate.

The application of the Fredholm integral equation to such a discrete-time Markov process may be limited because to obtain expressions of the probability and density functions, several conditions on transition probability densities must be satisfied. As Dibu et al. [5] have applied the Fredholm integral equations to MAP processes, the general method of Fredholm integral equations to other stochastic processes may exist. Further extensions of the equations to the general stochastic processes are left for future research.

References

[1] K. J. Arrow, T. Harris, and J. Marschak, Optimal inventory policy, *Econometrica: Journal of the Econometric Society*, 19(1951), 250–272.

[2] A. Daddi-Moussa-Ider, B. Kaoui and H. Löwen, Axisymmetric flow due to a Stokeslet near a finite-sized elastic membrane. *Journal of the Physical Society of Japan*, 88 (2019), 054401, arXiv:1901.04485. doi:10.7566/JPSJ.88.054401.

[3] A. Daddi-Moussa-Ider, Asymmetric stokes flow induced by a transverse point force acting near a finite-sized elastic membrane. *Journal of the Physical Society of Japan*, 89(2020), 124401, arXiv:2006.14375. doi:10.7566/JPSJ.89.124401.

[4] A. Doucet, A. M. Johansen and V. B. Tadic, On solving integral equations using Markov chain Monte Carlo methods. *Applied Mathematics and Computation*, 216 (2010), 2869–2880.

[5] A. S. Dibu, M. J. Jacob, A. D. Papaioannou and L Ramsden, Delayed capital injections for a risk process with Markovian arrivals. *Methodology and Computing in Applied Probability* 23 (2021), 1057-1076.

[6] A.J. Fabens, The solution of queueing and inventory models by semi-Markov processes, *Journal of the Royal Statistical Society: Series B (Methodological)*, 23(1961), 113–127.

[7] W. Feller, An Introduction to Probability Theory and Its Applications, 2nd edition, John Wiley &Sons (1971).

[8] C. Fuh, S. Fuh, Y. Liu and C. Wang, Renyi divergence in general hidden Markov models. arxiv.org/abs/2106.01645v1 (2021).

[9] P. Guo, Numerical simulation for Fredholm integral equation of the second kind. *Journal of Applied Mathematics and Physics*, 8(2020), 2438-2446.

[10] J.M. Gutiérrez, M.Á. Hernández-Verón, E. Martínez, Improved iterative solution of linear Fredholm integral equations of second kind via inverse-free iterative schemes. *Mathematics,* 8 (2020), 1747.

[11] R. Karim and K. Nakade, A Markovian production-inventory system with consideration of random quality disruption*. Journal of Advanced Mechanical Design Systems and Manufacturing*, 14(5) (2020), 1-18.

[12] T. Komorowski, and T. Szarek, On ergodicity of some Markov processes. *Annals of Probability,* 38 (2010), 1401–1443.

[13] C. Lindemann and A. Thümmler, Transient analysis of deterministic and stochastic Petri nets with concurrent deterministic transitions. *Performance Evaluation*, 36–37(1999), 35–54.

[14] M. A. Mohammad, Numerical solution of Fredholm integral equations of the second kind based on tight framelets generated by the oblique extension principle. *Symmetry,* 11(7) (2019) 854, DOI: 10.3390/sym11070854.

[15] L. Ramsden and A. Papaioannou, On the time to ruin for a dependent delayed capital injection risk model. *Applied Mathematics and Computation*, 352 (2019), 119-135.

[16] M. D. Raisinghania, Integral equations and boundary value problems, 1st edition. S. Chand & Company LTD (2007).

[17] Y. Tian, Markov chain Monte Carlo method to solve Fredholm integral equations. *Thermal science*, 22(4) (2018), 1673-1678.

[18] W. Werner, On the spatial Markov property of soups of unoriented and oriented loops. In Séminaire de Probabilités XLVIII, edited by Catherine Donati-Martin, Antoine Lejay, and Alain Rouault, 2016, 481–503. Charm: Springer.

[19] H. ZhiMin, Y. ZaiZai and C. JianRui, Monte Carlo method for solving the Fredholm integral equations of the second kind. *Transport Theory and Statistical Physics*, 41(2012), 513–528.

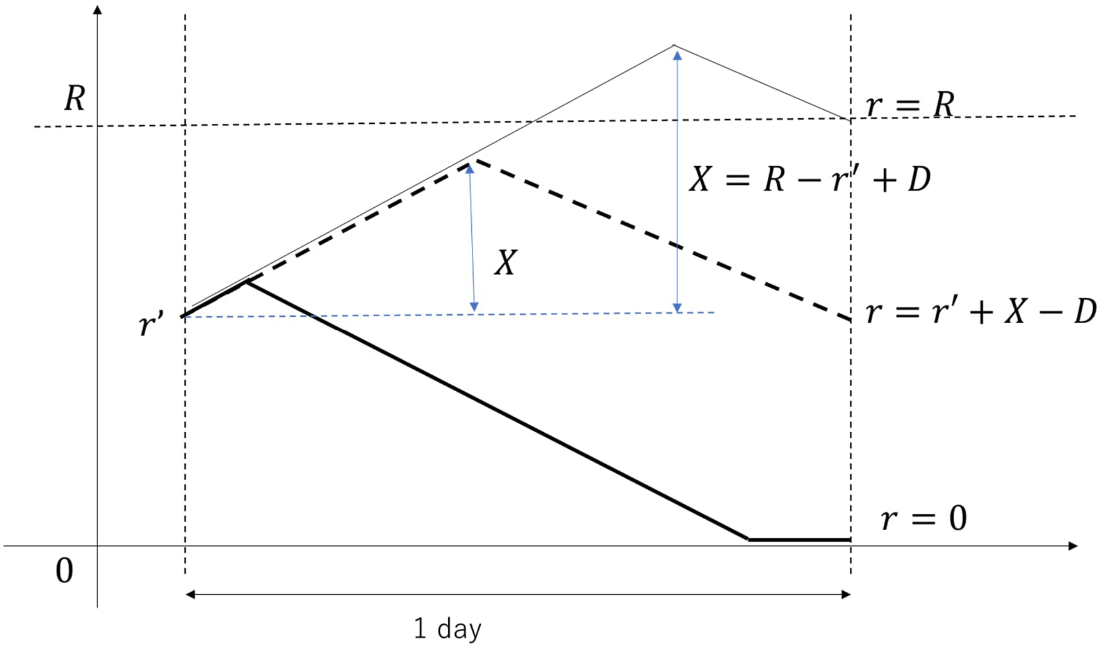

Fig.1 EPQ model

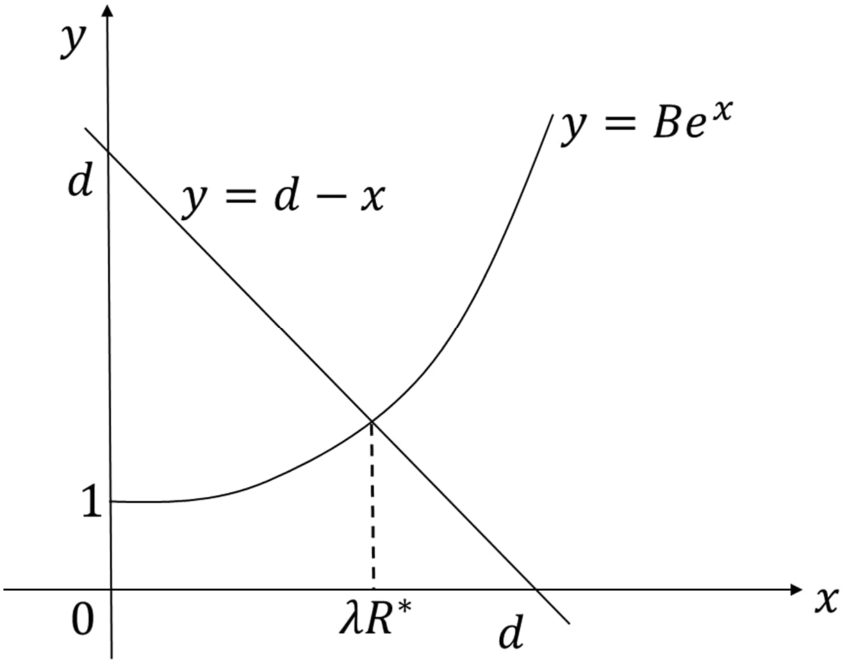

Fig. 2 λR^* is the intersecting point of two lines.

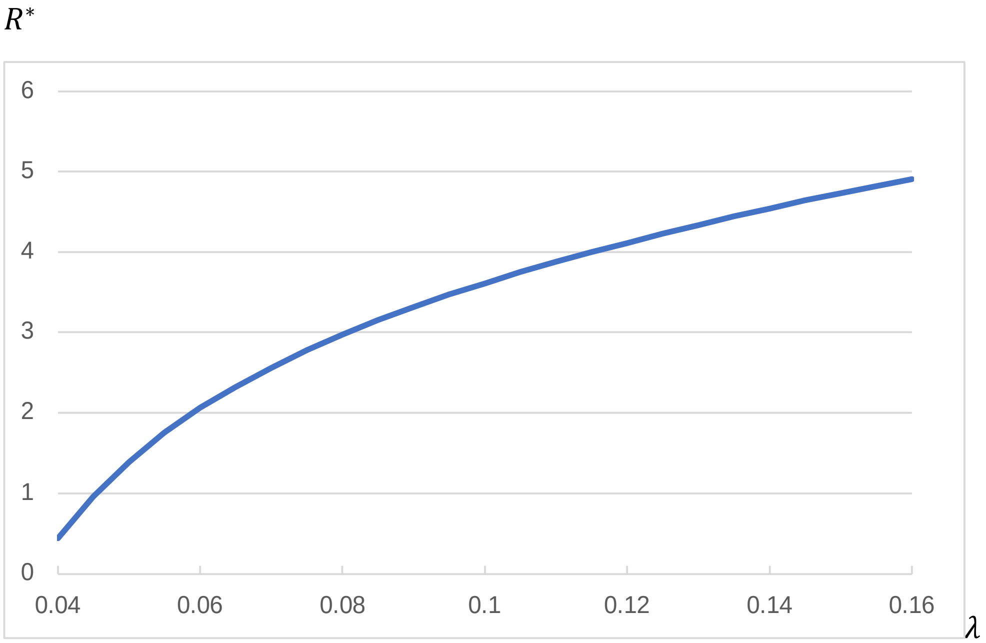

Fig.3 The relationship between optimal bounds and the arrival rate