RAIRO Operations Research
Will be set by the publisher

OPTIMALITY CONDITIONS AND DUALITY FOR MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING PROBLEMS ON HADAMARD MANIFOLDS USING GENERALIZED GEODESIC CONVEXITY

Balendu Bhooshan Upadhyay[1], Arnav Ghosh[1], Priyanka Mishra[1] and Savin Treanţă[2]

Abstract. This paper deals with multiobjective semi-infinite programming problems on Hadamard manifolds. We establish the sufficient optimality criteria of the considered problem under generalized geodesic convexity assumptions. Moreover, we formulate the Mond-Weir and Wolfe type dual problems and derive the weak, strong and strict converse duality theorems relating the primal and dual problems under generalized geodesic convexity assumptions. Suitable examples have also been given to illustrate the significance of these results. The results presented in this paper extend and generalize the corresponding results in the literature.

Keywords: Semi-infinite programming; Multiobjective optimization; Optimality; Duality; Hadamard manifolds

Mathematics Subject Classification. 90C34, 90C46, 90C48, 90C29, 58A05, 58C05, 49K27

May 29, 2022.

[1] Department of Mathematics, Indian Institute of Technology Patna, India;
e-mail: bhooshan@iitp.ac.in & arnav_2021ma09@iitp.ac.in & priyanka.iitp14@gmail.com
[2] Department of Applied Mathematics, University Politehnica of Bucharest, 060042 Bucharest, Romania;
e-mail: savin.treanta@upb.ro
[2] Academy of Romanian Scientists, Ilfov 3, 050044 Bucharest, Romania.

1. Introduction

In theory of optimization, semi-infinite programming is the class of mathematical programming problems that deals with finitely many decision variables and in which the feasible set is defined by infinitely many constraints. The concepts and mathematical theory of semi-infinite programming were conceived by Haar [24]. The term 'semi-infinite programming' was later coined by Charnes et al. [11] in 1962. Semi-infinite programming has a very wide range of applications in various practical problems of mathematical physics, game theory, engineering design, etc., see [12, 13, 16, 22, 23, 27, 29, 45, 50, 54] and the references cited therein.

Although semi-infinite programming over finite or infinite dimensional Banach spaces has been extensively studied, it is observed that a lot of programming problems that arise in various real life applications require the problem to be formulated on Riemannian manifolds. One of the very first attempts in this direction is due to Ekeland [19], who discussed applications of variational principles on Riemannian manifolds. The generalizations of optimization methods from Euclidean space to Riemannian manifolds have important advantages. For example, constrained optimization problems can be viewed as unconstrained problems from the Riemannian geometry perspective. Moreover, nonconvex optimization problems can be converted to convex optimization problems through introduction of a suitable Riemannian metric (see for instance, [17, 42, 43]). Some results on convex optimization problems were extended to Riemannian manifolds by Rapcsák [46] and Udrişte [59] by introducing a generalization of convexity notion, namely, geodesic convexity. Further, Udrişte [59] introduced the notions of geodesic pseudoconvex and quasiconvex functions in Riemannian manifold setting. Constrained optimization problems and weak sharp minimizers on Hadamard manifold were discussed by Li et al. [31]. Recently, many authors have generalized various other notions and concepts of optimization of $\mathbb{R}^n$ to Riemannian manifolds; see for instance, [1, 4, 6, 7, 8, 9, 10, 41, 53, 57, 58] and the references cited therein.

Optimality and duality conditions play a very crucial role in optimization theory. Duality theory is important to understand the nature of the original (primal problem) from the perspective of a dual problem. Many authors have developed many interesting results on optimality and duality in $\mathbb{R}^n$, see for instance, [20, 21, 32, 38, 54, 63] and the references cited therein. Two of the most important types of dual of a primal problem are Mond-Weir dual problem [60] and Wolfe dual problem [61], which have been referred to in this paper.

The optimality conditions for nonlinear programming problems on Riemannian manifolds were studied by Yang et al. in [62]. Bergmann and Herzog [8] developed the intrinsic formulation of Karush-Kuhn-Tucker conditions and constraint qualification on smooth manifolds. The necessary and sufficient optimality conditions for vector equilibrium problems on Hadamard manifolds have been discussed by Ruiz-Garzón et al. in [47]. Characterizations of solution sets of convex optimization problems in Riemannian manifolds were investigated by Barani and Hosseini in [4]. Further, Chen [15] studied the Karush-Kuhn-Tucker type optimality criteria for interval valued objective function on Hadamard manifolds. Optimality and duality for multiobjective semi-infinite programming on Hadamard manifolds was investigated by Tung and Tam in [56].

Motivated by the works of [15, 56] and the references cited therein, we consider a class of multiobjective semi-infinite programming problems on Hadamard manifold. We establish Karush-Kuhn-Tucker type sufficient optimality criteria for the considered problem under generalized geodesic convexity assumptions. Moreover, we formulate the Mond-Weir and Wolfe type dual problems and establish weak, strong and strict converse duality theorems relating the primal and the dual problems under generalized geodesic convexity assumptions. The results presented in this paper extend and generalize some known results in the literature to a more general space, namely, Hadamard manifold, as well as to more general classes of generalized geodesic convex functions. In particular, the results of this paper generalize the corresponding results of Tung and Tam [56] to a more general class of generalized geodesic convex function. Moreover, the results of the paper generalize some other well known results in $\mathbb{R}^n$, see for instance, [3, 34, 35, 36].

The paper is organized as follows. In Section 2, we recall the basic notions of Riemannian and Hadamard manifolds, that will be used in the sequel. Moreover, we consider a multiobjective semi-infinite programming problem on a Hadamard manifold. In Section 3, we establish sufficient optimality criteria of an efficient solution of the considered problem under generalized geodesic convexity assumptions. In Section 4, we formulate the Mond-Weir and Wolfe type dual problems for the considered primal problem. Moreover, we derive weak, strong and strict converse duality theorems relating the primal and the dual problems under generalized geodesic convexity assumptions.

2. Notations and mathematical Preliminaries

Throughout the paper, we use the standard notation $\mathbb{R}^n$ to denote the n-dimensional Euclidean plane. The nonnegative orthant of $\mathbb{R}^n$ is denoted by

$$\mathbb{R}^n_+ := \{(x_1, x_2, ..., x_n) : x_k \geq 0, \forall k = 1, 2, ..., n\}.$$

For a non empty infinite set J, the linear space $\mathbb{R}^{|J|}$ is defined as follows:

$$\mathbb{R}^{|J|} := \{\lambda = (\lambda_j)_{j\in J} : \lambda_j = 0 \text{ for all } j \in J, \text{ except } \lambda_j \neq 0 \text{ for finitely many } j \in J\}.$$

The positive cone of $\mathbb{R}^{|J|}$, denoted by $\mathbb{R}^{|J|}_+$, is defined as

$$\mathbb{R}^{|J|}_+ := \{\lambda = (\lambda_j)_{j\in J} \in \mathbb{R}^{|J|} : \lambda_j \geq 0, \forall j \in J\}.$$

The standard inner product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$. For any $x, y \in \mathbb{R}^n$, we use the following notations

$$x \prec y \iff x_i < y_i, \quad \forall i = 1, 2, ..., n.$$

$$x \preceq y \iff \begin{cases} x_i \leq y_i, & \text{for all } i = 1, 2, ..., n; i \neq k, \\ x_k < y_k, & \text{for at least one } k \in \{1, 2, .., n\}. \end{cases}$$

The notation $x \not\prec y$ (respectively, $x \not\preceq y$) indicates the negation of $x \prec y$ (respectively, $x \preceq y$).
If $\mathbb{E}$ is a m-dimensional linear subspace of $\mathbb{R}^n$, then $\mathbb{E}$ inherits the inner product from $\mathbb{R}^n$, denoted by $\langle \cdot, \cdot \rangle_{\mathbb{E}} = \langle \cdot, \cdot \rangle$. Moreover, the topology in $\mathbb{R}^n$ is induced to $\mathbb{E}$. One can further obtain a natural isometry between $\mathbb{R}^n$ and $\mathbb{E}$ (see for instance, [9]).
For a subset $S \subset \mathbb{E}$, the closure and convex hull of S in $\mathbb{E}$ is denoted by $\mathrm{cl}(S)$ and $\mathrm{co}(S)$, respectively. The positive conic hull of S, denoted by $\mathrm{pos}(S)$, is the convex cone containing the origin generated by $S \subset \mathbb{E}$, and is defined as

$$\mathrm{pos}(S) := \{\sum_{i=1}^{n} \alpha_i x_i, \alpha_i \geq 0, x_i \in S, n \in \mathbb{N}\},$$

where, $\mathbb{N}$ denotes the set of all natural numbers. The negative polar cone of S, denoted by S^-, is defined by

$$S^- := \{x \in \mathbb{E} : \langle x, y \rangle \leq 0, \forall y \in S\}.$$

For any two Euclidean spaces $\mathbb{E}_1, \mathbb{E}_2$, a map $\phi : \mathbb{E}_1 \to \mathbb{E}_2$ is said to be of class C^1 if ϕ is continuously differentiable. Similarly, $\phi : \mathbb{E}_1 \to \mathbb{E}_2$ is said to be of class C^∞ if ϕ is infinitely continuously differentiable.
Now, we recall some fundamental concepts and definitions of Riemannian and Hadamard manifolds (see for instance, [2, 9, 25, 28, 30]).

Let us consider that $\mathscr{H}$ be a topological space. Then $\mathscr{H}$ is said to be topological n-manifold or a topological manifold of dimension n if $\mathscr{H}$ is Hausdorff, second-countable, and each point of $\mathscr{H}$ is contained in some neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. Any pair (U, ϕ), where U is an open set in $\mathscr{H}$, and ϕ is a homeomorphism from U to some open set in $\mathbb{R}^n$ is called a chart or a co-ordinate chart on $\mathscr{H}$. For any two charts (U, ϕ) and (V, ψ), such that $U \cap V$ is non empty, the composite map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is called the transition map from ϕ to ψ. Two charts (U, ϕ) and (V, ψ) are said to be smoothly compatible if either $U \cap V$ is empty or the transition map $\psi \circ \phi^{-1}$ is infinitely continuously differentiable. A collection of charts such that the corresponding open sets cover $\mathscr{H}$ is called an atlas. An atlas $\mathcal{A}$ for $\mathscr{H}$ is said to be smooth if any two charts in $\mathcal{A}$ are smoothly compatible with each other. A smooth atlas $\mathcal{A}$ on $\mathscr{H}$ is said to be maximal if it is not properly contained in any larger smooth atlas. A maximal smooth atlas on $\mathscr{H}$ is called a smooth structure. A smooth manifold is a pair $(\mathscr{H}, \mathcal{A})$ where $\mathscr{H}$ is a topological manifold and $\mathcal{A}$ is a smooth structure on

$\mathscr{H}$.
For an element $p \in \mathscr{H}$, a curve $\gamma : (-\epsilon, \epsilon) \to \mathscr{H}$ is said to be of class C^1 about the point p if $\gamma(0) = p$, and $\phi \circ \gamma$ is of class C^1 for any chart (U, ϕ) about the point p. Let $\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \to \mathscr{H}$ be any two C^1 curves about p. Then γ_1 and γ_2 are said to be equivalent if and only if there exists some chart (U, ϕ) about the point p, such that

$$(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0).$$

A tangent vector to $\mathscr{H}$ at the point p is any equivalence class of C^1 curves through p on $\mathscr{H}$ modulo the equivalence relation defined above. The set of all tangent vectors at the point p in $\mathscr{H}$ is termed as the tangent space to $\mathscr{H}$ at p and is denoted by the symbol $T_p\mathscr{H}$.
A Riemannian metric on a smooth manifold $\mathscr{H}$ is defined as a 2-tensor field $\mathscr{G}$, such that $\mathscr{G}$ is symmetric and positive definite. An inner product is induced on every tangent space $T_p\mathscr{H}$ by a Riemannian metric and this is denoted by $\mathscr{G}(x, y) = \langle x, y \rangle_p$ for all $x, y \in T_p\mathscr{H}$. A smooth manifold together with a given Riemannian metric is called a Riemannian manifold. The exponential map $\exp_p : T_p\mathscr{H} \to \mathscr{H}$ is defined by $\exp_p(v) = \gamma_{p,v}(1)$ for any $v \in T_p\mathscr{H}$, where $\gamma_{p,v}$ is the geodesic starting at p with a velocity v.

A Riemannian manifold $\mathscr{H}$ is said to be geodesic complete if for every $p \in \mathscr{H}$, the exponential map $\exp_p(v)$ is defined for all $v \in T_p\mathscr{H}$. A complete, simply connected Riemannian manifold with nonpositive sectional curvature everywhere is called a Hadamard manifold. Henceforth, we shall use to the symbol $\mathscr{H}$ to denote a Hadamard manifold, unless otherwise specified.
The following theorem, known as Hadamard-Cartan theorem, is from Sakai [48] (Theorem 4.1, Page 221).

Theorem 2.1. *Let $\mathscr{H}$ be a Hadamard manifold. Then for every $p \in T_p\mathscr{H}$, the exponential map $exp_p : T_p\mathscr{H} \to \mathscr{H}$ is a diffeomorphism with the inverse map $exp_p^{-1} : \mathscr{H} \to T_p\mathscr{H}$ satisfying $exp_p^{-1}(p) = 0_p$. Moreover, for any $x \in \mathscr{H}$ there exists a unique minimal geodesic $\gamma_{p,x} : [0,1] \to \mathscr{H}$ satisfying $\gamma_{p,x}(t) = exp_p(t exp\,_p^{-1}(x))$.*

The contingent cone for a subset of a Hadamard manifold is defined as follows.

Definition 2.2. Let $S \subseteq \mathscr{H}$ and $p \in \mathrm{cl}(S)$. The contingent cone of S at p, denoted by $\mathscr{T}(S, p)$ is defined by

$$\mathscr{T}(S,p) = \{v \in T_p\mathscr{H} : \exists t_k \downarrow 0, \exists v_k \in T_p\mathscr{H}, v_k \to v, \forall k \in \mathbb{N}, \exp_p(t_k v_k) \in S\}.$$

The following definitions of geodesic convex sets and geodesic convex functions on a Riemannian manifold are from Udrişte [59] (Page 57) and Rapcsák [46] (Definition 6.1.2, Page 64), respectively.

Definition 2.3. Let us consider that $\mathscr{H}$ be a Riemannian manifold. Then,
(i) A subset S of $\mathscr{H}$ is called a geodesic convex set in $\mathscr{H}$, if for every pair of

distinct points $x, y \in S$ and for any geodesic $\gamma_{x,y} : [0,1] \to \mathscr{H}$ joining x to y, we have

$$\gamma_{x,y}(t) \in S, \quad \forall t \in [0,1].$$

(ii) Let S be a geodesic convex subset of $\mathscr{H}$ and $f : S \to \mathbb{R}$ be a function on S. Then, the function f is said to be geodesic convex at $x \in S$, if for any point $u \in S$ and for any geodesic $\gamma_{x,u} : [0,1] \to \mathscr{H}$ joining x to u, we have

$$f\Big(\gamma_{x,u}(t)\Big) \leq tf(x) + (1-t)f(u), \quad \forall t \in [0,1].$$

When the preceding inequality is strict, for $x \neq u$ and $t \in (0,1)$, the function f is said to be geodesic strictly convex at $x \in S$.
In particular, if $\mathscr{H}$ is a Hadamard manifold, then f is geodesic convex at x if and only if the following holds

$$f\bigg(\exp_x\Big(t\exp_x^{-1}(u)\Big)\bigg) \leq tf(x) + (1-t)f(u), \quad \forall u \in S, \forall t \in [0,1].$$

When the preceding inequality is strict, for $x \neq u$ and $t \in (0,1)$, the function f is said to be geodesic strictly convex at $x \in S$.

The following theorems from Rapcsák [46] (Theorem 6.3.1 and Corollary 6.3.1, Page 76-77) enable us to check for geodesic convexity more efficiently.

Theorem 2.4. *Let $S \subset \mathscr{H} \subset \mathbb{R}^n$ be an open geodesic convex set, and $f : S \to \mathbb{R}$ be a twice continuously differentiable function. Then, f is geodesic convex on S if and only if the geodesic Hessian (or second covariant derivative), denoted by $H_u^g f(x(u))$, and defined as follows*

$$H_u^g f(x(u)) = J(x(u))^T H_x f(x(u)) Jx(u) + \nabla_x f(x(u))(Hx(u) - Jx(u)\Gamma(u)),$$

$$Hx(u) = \begin{pmatrix} Hx_1(u) \\ \vdots \\ Hx_n(u) \end{pmatrix}, \quad Jx(u) = \left(\frac{\partial x}{\partial u}\right),$$

where $Hx(u)$ is Hessian matrix, $Jx(u)$ is the matrix of first partial derivatives, $H_x f(x(u))$ denotes the Hessian matrix of the function f by x at $x(u)$, and $\Gamma(u)$ is the matrix of second Christoffel symbols with respect to the Riemannian metric of $\mathscr{H}$, is positive semidefinite at all the points of each geodesic convex coordinate neighbourhood $x(u)$ of S.

Theorem 2.5. *Let $S \subset \mathscr{H} = \mathbb{R}^n$ be an open geodesic convex set, and $f : S \to \mathbb{R}$ be a twice continuously differentiable function. Then, f is geodesic convex on S if and only if the following matrix*

$$H^g f(x) = \nabla^2 f(x) + \nabla f(x)\Gamma,$$

where Γ is the matrix of second Christoffel symbols with respect to the Riemannian metric of $\mathbb{R}^n, \nabla f(x)$ and $\nabla^2 f(x)$ are the (Euclidean) gradient and (Euclidean) Hessian of the function f at x in the usual sense, is positive semidefinite at all the points of each geodesic convex coordinate neighbourhood $x(u)$, with $x : U \subset \mathbb{R}^k \to \mathbb{R}^n$, of S.

The following definitions of geodesic pseudoconvex and quasiconvex functions on Hadamard manifolds are taken from Definition 2.1 of Barani [5].

Definition 2.6. Let $S \subseteq \mathscr{H}$ be a geodesic convex set. Then
(i) A map $f : S \to \mathbb{R}$ is said to be geodesic pseudoconvex at $y \in S$, if for any arbitrary point $x \in S$, we have

$$\left\langle \operatorname{grad} f(y), \exp_y^{-1}(x) \right\rangle_y \geq 0 \implies f(x) - f(y) \geq 0.$$

A map $f : S \to \mathbb{R}$ is said to be geodesic strictly pseudoconvex at $y \in S$, if for any arbitrary point $x \in S$, $x \neq y$, we have

$$\left\langle \operatorname{grad} f(y), \exp_y^{-1}(x) \right\rangle_y \geq 0 \implies f(x) - f(y) > 0.$$

Equivalently, a map $f : S \to \mathbb{R}$ is said to be geodesic pseudoconvex at $y \in S$, if for any arbitrary point $x \in S$, we have

$$f(x) - f(y) < 0 \implies \left\langle \operatorname{grad} f(y), \exp_y^{-1}(x) \right\rangle_y < 0.$$

A map $f : S \to \mathbb{R}$ is said to be geodesic strictly pseudoconvex at $y \in S$, if for any arbitrary point $x \in S$, $x \neq y$, we have

$$f(x) - f(y) \leq 0 \implies \left\langle \operatorname{grad} f(y), \exp_y^{-1}(x) \right\rangle_y < 0.$$

(ii) A map $f : S \to \mathbb{R}$ is said to be geodesic quasiconvex at $y \in S$, if for any arbitrary point $x \in S$, we have

$$f(x) - f(y) \leq 0 \implies \left\langle \operatorname{grad} f(y), \exp_y^{-1}(x) \right\rangle_y \leq 0.$$

Equivalently, a map $f : S \to \mathbb{R}$ is said to be geodesic quasiconvex at $y \in S$, if for any arbitrary point $x \in S$, we have

$$\left\langle \operatorname{grad} f(y), \exp_y^{-1}(x) \right\rangle_y > 0 \implies f(x) - f(y) > 0.$$

Remark 2.7. (1) If S be a convex subset of $\mathscr{H} = \mathbb{R}^n$, then, $\operatorname{grad} f(y) = \nabla f(y)$, and $\exp_y^{-1}(x) = x - y$. Then, the above definitions of geodesic (strict) pseudoconvexity and geodesic quasiconvexity reduce to the standard definitions of differentiable (strict) pseudoconvex and quasiconvex functions given in Mangasarian [34] (Page 146) for $\mathbb{R}^n$.
(2) In view of Definition 10.1 in Udrişte [59] and Definition 13.2.1 in Rapcsák [46], if $f : S \to \mathbb{R}$ in the above definitions is a geodesic convex function, then it is also a geodesic pseudoconvex and a geodesic quasiconvex function.

For more details on generalized geodesic convex functions on Hadamard manifolds, we refer the reader to [14, 37, 51, 52] and the references cited therein.
The following theorem is from Shahi and Mishra [49] (see Theorem 3.2 (a)). We present a proof of the theorem here for the sake of convenience of the readers.

Theorem 2.8. *Let us assume that the function $\theta(u) = \frac{\phi(u)}{\psi(u)}$, where ϕ and ψ are smooth functions on an open geodesic convex set S of a Riemannian manifold $\mathscr{H}$. If ϕ be a geodesic convex function and ψ be a positive affine function, then θ be a geodesic pseudoconvex function.*

Proof. Let G denote the set of all geodesics connecting $u, v \in S$. Since ϕ is geodesic convex, then $\forall u, v \in S$, $\forall \bar{\gamma}_{uv} \in G$, we have

$$\phi\left(\bar{\gamma}_{uv}(t)\right) \leq t\phi(v) + (1-t)\phi(u), \quad \forall t \in [0,1]. \tag{1}$$

Since ψ is affine, then $\forall u, v \in S$, $\forall \bar{\gamma}_{uv} \in G$, we have

$$\psi\left(\bar{\gamma}_{uv}(t)\right) = t\psi(v) + (1-t)\psi(u), \quad \forall t \in [0,1]. \tag{2}$$

From (1) and (2), we get that,

$$\frac{\phi\left(\bar{\gamma}_{uv}(t)\right)}{\psi\left(\bar{\gamma}_{uv}(t)\right)} \leq \frac{t\phi(v) + (1-t)\phi(u)}{t\psi(v) + (1-t)\psi(u)}, \quad \forall t \in [0,1]. \tag{3}$$

Let us assume that $\forall u, v \in S$,

$$\dot{\bar{\gamma}}_{uv}\left(\frac{\phi}{\psi}\right)(u) \geq 0. \tag{4}$$

Then, it follows that

$$\begin{aligned} &\lim_{t\to 0}\left[\frac{\frac{\phi}{\psi}\left(\bar{\gamma}_{uv}(t)\right) - \frac{\phi}{\psi}(u)}{t}\right] \geq 0, \\ \text{or } &\lim_{t\to 0}\frac{1}{t}\left[\frac{t\phi(v) + (1-t)\phi(u)}{t\psi(v) + (1-t)\psi(u)} - \frac{\phi(u)}{\psi(u)}\right] \geq 0. \end{aligned} \tag{5}$$

Therefore, we get the following

$$\lim_{t\to 0}\left[\frac{t\psi(u)\phi(v)+(1-t)\phi(u)\psi(u)-t\phi(u)\psi(v)-(1-t)\phi(u)\psi(u)}{t\psi(u)\{t\psi(v)+(1-t)\psi(u)\}}\right]\geq 0,$$

an indeterminate form as $\left(\frac{0}{0}\right)$. Using L'Hospital's rule, we obtain

$$\begin{aligned}
&\lim_{t\to 0}\left[\frac{\psi(u)\phi(v)-\phi(u)\psi(u)-\phi(u)\psi(v)+\phi(u)\psi(u)}{2t\psi(u)\psi(v)+(1-2t)\psi^2(u)}\right]\geq 0\\
&\text{or}\ \left[\frac{\psi(u)\phi(v)-\phi(u)\psi(v)}{\psi^2(u)}\right]\geq 0\\
&\text{or}\ \left[\frac{\phi(v)}{\psi(u)}-\frac{\phi(u)\psi(v)}{\psi^2(u)}\right]\geq 0\\
&\text{or}\ \ \phi(v)\geq\frac{\phi(u)\psi(v)}{\psi(u)},
\end{aligned}\tag{6}$$

since, ψ is positive. Thus, we get

$$\frac{\phi(v)}{\psi(v)}\geq\frac{\phi(u)}{\psi(u)}.$$

Therefore, $\frac{\phi}{\psi}$ is a geodesic pseudoconvex function on S. This completes the proof. □

In this paper, we consider the following multiobjective semi-infinite programming problem on Hadamard manifold:

$$\begin{aligned}
\text{(MSIP)}\quad &\text{Minimize } f(x)=(f_1(x),...,f_m(x)),\\
&\text{subject to } \ g_j(x)\leq 0,\quad j\in J,
\end{aligned}$$

where, $f_i : S\to\mathbb{R}$, $(i\in I=\{1,2,...,m\})$, $g_j : S\to\mathbb{R}$ $(j\in J)$, are smooth functions defined on an open geodesic convex set $S\subset\mathscr{H}$. The index set J is arbitrary. The feasible set of the problem (MSIP), denoted by F, is defined by

$$F:=\{x\in S : g_j(x)\leq 0, \forall j\in J\}.$$

The index set of all active inequality constraints at a feasible point $x\in F$, denoted by $L(x)$, is defined as

$$L(x):=\{j\in J : g_j(x)=0\}.$$

For any feasible point $x\in F$, we denote the set of all active contraint multipliers at $x\in F$ by $\mathscr{A}(x)$, that is

$$\mathscr{A}(x):=\{\lambda\in\mathbb{R}^{|J|}_+ : \lambda_j g_j(x)=0, \forall j\in J\}.$$

Now, we recall the notions of efficient solution and weakly efficient solution of (MSIP) (see for instance, [33, 56]).

Definition 2.9. A point $\bar{x} \in F$ is said to be an efficient solution of the problem (MSIP), if there exists no other point $x \in F$, such that

$$f(x) \preceq f(\bar{x}).$$

Definition 2.10. A point $\bar{x} \in F$ is said to be a weakly efficient solution of the problem (MSIP), if there exists no other point $x \in F$, such that

$$f(x) \prec f(\bar{x}).$$

Now, we state a constraint qualification analogous to Abadie constraint qualification for (MSIP) from Tung and Tam [56].

Definition 2.11. Let $\bar{x} \in F$. Then, the Abadie constraint qualification (ACQ) is said to be satisfied at the point $\bar{x}$, if

$$\left(\bigcup_{j \in L(\bar{x})} \operatorname{grad} g_j(\bar{x}) \right)^{-} \subseteq \mathscr{T}(\bar{x}, F),$$

and the set pos $\bigcup_{j \in L(\bar{x})} \operatorname{grad} g_j(\bar{x})$ is closed.

3. Optimality conditons

In this section, we derive a sufficient optimality criteria for (MSIP) using generalized geodesic convex functions.
To begin with, we state the Karush-Kuhn-Tucker type necessary optimality criteria for (MSIP) from Tung and Tam [56].

Theorem 3.1. *Let $\bar{x}$ be a weakly efficient solution of (MSIP) such that Abadie constraint qualification (ACQ) is satisfied at $\bar{x}$. Then, there exist some $\alpha \in \mathbb{R}^m_+$, satisfying $\sum_{i \in I} \alpha_i = 1$, and some $\lambda \in \mathscr{A}(\bar{x})$, such that the following equation holds*

$$\sum_{i \in I} \alpha_i \operatorname{grad} f_i(\bar{x}) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(\bar{x}) = 0.$$

Now, we derive the sufficient optimality criteria of the problem (MSIP) using generalized geodesic convexity assumptions.

Theorem 3.2. *Let $\bar{x} \in F$ be an arbitrary feasible point. Let us assume that there exist some $\alpha \in \mathbb{R}^m_+$, satisfying $\sum_{i \in I} \alpha_i = 1$, and some $\lambda \in \mathscr{A}(\bar{x})$, such that the following equation holds*

$$\sum_{i \in I} \alpha_i \operatorname{grad} f_i(\bar{x}) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(\bar{x}) = 0. \tag{7}$$

Then the following statements are true.

(i) If f_i is geodesic pseudoconvex at $\bar{x} \in F$ for all $i \in I$, and g_j is geodesic quasiconvex at $\bar{x} \in F$ for all $j \in J$, then $\bar{x}$ is a weakly efficient solution of (MSIP).

(ii) If f_i is geodesic strictly pseudoconvex at $\bar{x} \in F$ for all $i \in I$, and g_j is geodesic quasiconvex at $\bar{x} \in F$ for all $j \in J$, then $\bar{x}$ is an efficient solution of (MSIP).

Proof. Since $\lambda \in \mathscr{A}(\bar{x})$, there exists a finite subset K of $L(\bar{x})$, such that

$$\lambda_j > 0, \quad \forall j \in K,$$
$$\lambda_j = 0, \quad \forall j \in L(\bar{x}) \setminus K.$$

From condition (7), we have

$$\sum_{i \in I} \alpha_i \operatorname{grad} f_i(\bar{x}) = -\sum_{j \in K} \lambda_j \operatorname{grad} g_j(\bar{x}). \tag{8}$$

Let $x \in F$ be an arbitrary feasible point. Then,

$$g_j(x) \leq 0 = g_j(\bar{x}), \quad \forall j \in K.$$

As $g_j(x)$ is geodesic quasiconvex at $\bar{x}$ on F for all $j \in J$, we have

$$g_j(x) \leq g_j(\bar{x}) \implies \Big\langle \operatorname{grad} g_j(\bar{x}), \exp{}_{\bar{x}}^{-1}(x) \Big\rangle_{\bar{x}} \leq 0, \quad \forall j \in K.$$

Since $\lambda \in \mathbb{R}_+^{|J|}$, it follows that

$$\sum_{j \in K} \lambda_j \Big\langle \operatorname{grad} g_j(\bar{x}), \exp{}_{\bar{x}}^{-1}(x) \Big\rangle_{\bar{x}} \leq 0. \tag{9}$$

From inequality (9) and equation (8), we have

$$\sum_{i \in I} \Big\langle \alpha_i \operatorname{grad} f_i(\bar{x}), \exp{}_{\bar{x}}^{-1}(x) \Big\rangle_{\bar{x}} = -\sum_{j \in K} \lambda_j \Big\langle \operatorname{grad} g_j(\bar{x}), \exp{}_{\bar{x}}^{-1}(x) \Big\rangle_{\bar{x}} \geq 0.$$

That is

$$\sum_{i \in I} \Big\langle \alpha_i \operatorname{grad} f_i(\bar{x}), \exp{}_{\bar{x}}^{-1}(x) \Big\rangle_{\bar{x}} \geq 0, \quad \forall x \in F. \tag{10}$$

(i) On the contrary, let us assume that $\bar{x}$ is not a weakly efficient solution of (MSIP). Then, there exists some $\hat{x} \in F$, such that

$$f_i(\hat{x}) < f_i(\bar{x}), \quad \forall i \in I.$$

As $f_i(x)$ is geodesic pseudoconvex at $\bar{x} \in F$ for all $i \in I$, we have

$$f_i(\hat{x}) < f_i(\bar{x}) \implies \left\langle \operatorname{grad} f_i(\bar{x}), \exp{}_{\bar{x}}^{-1}(\hat{x}) \right\rangle_{\bar{x}} < 0, \quad \forall i \in I. \tag{11}$$

Combining inequality (11) with $\alpha \in \mathbb{R}_+^m$, $\sum_{i\in I} \alpha_i = 1$, we obtain

$$\sum_{i\in I} \left\langle \alpha_i \operatorname{grad} f_i(\bar{x}), \exp{}_{\bar{x}}^{-1}(\hat{x}) \right\rangle_{\bar{x}} < 0,$$

which is a contradiction to (10). This proves that $\bar{x}$ is a weakly efficient solution of (MSIP).
(ii) On the contrary, let us assume $\bar{x}$ is not an efficient solution of (MSIP). Then, there exists some $\hat{x} \in F$, such that

$$\begin{aligned} &f_i(\hat{x}) \leq f_i(\bar{x}), \quad \forall i \in I, i \neq p, \\ &f_p(\hat{x}) < f_p(\bar{x}), \quad \text{for at least one } p \in I. \end{aligned}$$

The above inequalities imply that $\hat{x} \neq \bar{x}$. Since, $f_i(x)$ is geodesic strictly pseudoconvex at $\bar{x} \in F$ for all $i \in I$, we have

$$\left\langle \operatorname{grad} f_i(\bar{x}), \exp{}_{\bar{x}}^{-1}(\hat{x}) \right\rangle_{\bar{x}} < 0, \quad \forall i \in I. \tag{12}$$

Combining inequality (12) with $\alpha \in \mathbb{R}_+^m$, $\sum_{i\in I} \alpha_i = 1$, we obtain

$$\sum_{i\in I} \alpha_i \left\langle \operatorname{grad} f_i(\bar{x}), \exp{}_{\bar{x}}^{-1}(\hat{x}) \right\rangle_{\bar{x}} < 0,$$

which is a contradiction to (10). This proves that $\bar{x}$ is an efficient solution of (MSIP).

□

The following example illustrates the significance of Theorem 3.2.

Example 3.3. Let us consider the Poincaré half-plane defined as follows

$$\mathscr{H} := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}.$$

Then, $\mathscr{H}$ is a Riemannian manifold equipped with the inner product (see for instance, Example 5, Page 2 in [59]), as follows

$$\langle u, v \rangle_x = \langle \mathscr{G}(x)u, v \rangle, \quad \forall u, v \in T_x\mathscr{H} = \mathbb{R}^2,$$

where $\mathscr{G} = \begin{bmatrix} \frac{1}{x_2^2} & 0 \\ 0 & \frac{1}{x_2^2} \end{bmatrix}$, and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^2$. Since the sectional curvature of $\mathscr{H}$ is -1, it is also a Hadamard manifold. The second Christoffel symbols are as follows:

$$\Gamma^{x_1} = \begin{pmatrix} 0 & -\frac{1}{x_2} \\ -\frac{1}{x_2} & 0 \end{pmatrix}, \quad \Gamma^{x_2} = \begin{pmatrix} \frac{1}{x_2} & 0 \\ 0 & -\frac{1}{x_2} \end{pmatrix}.$$

Let us consider the following open geodesic convex set on the Hadamard manifold $\mathscr{H}$ as follows:

$$S := \{x \in \mathscr{H} : x_1 > -\frac{1}{2}\}.$$

We consider the following semi-infinite programming problem on the Hadamard Manifold $\mathscr{H}$.

$$\begin{aligned} (P_1) \quad & \text{Minimize } f(x) = \Big(f_1(x), f_2(x)\Big) := \left(\frac{x_1^2}{2x_2} + \frac{x_2}{2}, \frac{\ln^2 \frac{x_1 + \frac{1}{2}}{x_2}}{2}\right), \\ & \text{subject to } \; g_j(x) := \frac{j}{x_2} - j - 1 \leq 0, \quad j \in J = [0,1]. \end{aligned}$$

Here $f_i, g_j : S \to \mathbb{R}$, $i = 1, 2$ are real valued functions and $j \in J$. The feasible set F of the problem is

$$\begin{aligned} F &= \{x \in S : x_2 \geq \frac{1}{2}\} \\ &= \{x \in \mathscr{H} : x_1 > -\frac{1}{2}, x_2 \geq \frac{1}{2}\}. \end{aligned}$$

Let us consider the point $\bar{x} = (0, \frac{1}{2}) \in F$. Then, it can be verified that

$$\mathscr{T}(\bar{x}, F) = \{v = (v_1, v_2) \in T_{\bar{x}}S : v_1, v_2 \geq 0\}.$$

Also, we have the following

$$\operatorname{grad} f_1(\bar{x}) = \left(0, \frac{1}{8}\right), \quad \operatorname{grad} f_2(\bar{x}) = (0, 0), \quad \operatorname{grad} g_j(\bar{x}) = (0, -j).$$

Then, it follows that

$$\bigcup_{j \in J(\bar{x})} \operatorname{grad} g_j(\bar{x}) = \{x \in T_{\bar{x}}S \mid x_1 = 0, -1 \leq x_2 \leq 0\},$$

and hence,

$$\left(\bigcup_{j \in J(\bar{x})} \operatorname{grad} g_j(\bar{x})\right)^{-} = \{x^* = (x_1^*, x_2^*) \in T_{\bar{x}}S \mid x_2^* \geq 0\} \subseteq \mathscr{T}(F, \bar{x}).$$

Further, it follows that

$$\text{pos} \bigcup_{j \in J(\bar{x})} \operatorname{grad} g_j(\bar{x}) = \{x \in T_{\bar{x}} S \mid x_1 = 0, x_2 \leq 0\}$$

is closed. That is, (ACQ) is satisfied at the feasible point $\bar{x} = (0, \frac{1}{2})$.

Let $\lambda : J \to \mathbb{R}$ be defined as follows

$$\lambda(j) = \begin{cases} \frac{1}{16}, & j = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then, there exist $\alpha = (\alpha_1, \alpha_2) = \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^2$, satisfying $\sum_{i=1}^{2} \alpha_i = 1$ and $\lambda \in \mathscr{A}(\bar{x})$, such that

$$\sum_{i=1}^{2} \alpha_i \operatorname{grad} f_i(\bar{x}) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(\bar{x}) = \frac{1}{2}(0, \frac{1}{8}) + \frac{1}{2}(0, 0) + \frac{1}{16}(0, -1) = (0, 0).$$

Now, we write $f_1(x)$ and $f_2(x)$ in the following manner

$$f_1(x) = \frac{\frac{x_1^2}{x_2} + x_2}{2},$$
$$f_2(x) = \frac{\ln^2 \frac{x_1 + \frac{1}{2}}{x_2}}{2}.$$

Now, we see that, the (hyperbolic) Hessian, or the second-order covariant derivative, of $\frac{x_1^2}{x_2} + x_2$ is given by

$$\begin{aligned}
&H^g\left(\frac{x_1^2}{x_2} + x_2\right) \\
&= \nabla^2\left(\frac{x_1^2}{x_2} + x_2\right) - \nabla\left(\frac{\frac{x_1^2}{x_2} + x_2}{2}\right)\Gamma \\
&= \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix} - \left(\frac{2x_1}{x_2}\begin{bmatrix} 0 & -\frac{1}{x_2} \\ -\frac{1}{x_2} & 0 \end{bmatrix} + \left(1 - \frac{x_1^2}{x_2^2}\right)\begin{bmatrix} \frac{1}{x_2} & 0 \\ 0 & -\frac{1}{x_2} \end{bmatrix}\right) \\
&= \begin{bmatrix} \frac{x_1^2 + x_2^2}{x_2^3} & 0 \\ 0 & \frac{x_1^2 + x_2^2}{x_2^3} \end{bmatrix}.
\end{aligned}$$

is a positive semidefinite matrix as all its eigen values are non negative. Thus, $\frac{x_1^2}{x_2} + x_2$ is geodesic convex. Similarly, it can be shown that $H^g\left(\ln^2 \frac{x_1 + \frac{1}{2}}{x_2}\right)$ is a positive semidefinite matrix. That is, $\ln^2 \frac{x_1+\frac{1}{2}}{x_2}$ is also a convex function. Thus, $f_1(x)$ and $f_2(x)$ are both ratios of geodesic convex functions and positive affine functions. Then, from Theorem 2.8, it follows that f_1, f_2 are geodesic pseudoconvex functions. Also, we have

$$H^g g_j(x) = \nabla^2 g_j(x) - \nabla g_j(x)\Gamma = \begin{bmatrix} \frac{j}{x_2^3} & 0 \\ 0 & \frac{j}{x_2^3} \end{bmatrix}.$$

Since the matrix $H^g g_j(x)$ is positive semidefinite, g_j is quasiconvex for all $j \in J$. This shows that the conditions of Theorem 3.2 (i) hold. It can be verified that $\bar{x}$ is a weakly efficient solution of the problem (P_1).

4. Duality

In this section, we formulate the Mond-Weir [60] and Wolfe [61] type dual problems for (MSIP) and establish weak, strong and strict converse duality theorems relating the primal problem (MSIP) and the dual problems under generalized geodesic convexity assumptions.

4.1. Mond-Weir duality

Let us consider that $u \in S \subset \mathscr{H}$, where S is an open geodesic convex set in $\mathscr{H}$, $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$ and $\lambda \in \mathbb{R}^{|J|}_+$. The Mond-Weir dual problem of (MSIP), denoted by $(\mathrm{MSID_{MW}})$, is formulated as follows:

$$\begin{aligned} (\mathrm{MSID_{MW}}) \quad & \text{Maximize } \widetilde{f}(u) := (f_1(u), f_2(u), ..., f_m(u)), \\ & \text{subject to } \sum_{i\in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j\in J} \lambda_j \operatorname{grad} g_j(u) = 0, \\ & \sum_{j\in J} \lambda_j g_j(u) \geq 0, \\ & u \in S, \alpha \in \mathbb{R}^m_+ \setminus \{0\}, \lambda \in \mathbb{R}^{|J|}_+. \end{aligned}$$

The feasible set of $\mathrm{MSID_{MW}}$, denoted by $\mathrm{F_{MW}}$, is given by

$$\begin{aligned} F_{MW} := \{(u, \alpha, \lambda) \in & S \times \mathbb{R}^m_+ \times \mathbb{R}^{|J|}_+ : \alpha \neq 0, \\ & \sum_{i\in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j\in J} \lambda_j \operatorname{grad} g_j(u) = 0, \sum_{j\in J} \lambda_j g_j(u) \geq 0\}. \end{aligned}$$

The following definitions of efficient solution and weakly efficient solution of the Mond-Weir dual problem $(\mathrm{MSID_{MW}})$ are from Tung and Tam [56].

Definition 4.1. Let $(\tilde{u}, \tilde{\alpha}, \tilde{\lambda}) \in F_{MW}$. Then, $(\tilde{u}, \tilde{\alpha}, \tilde{\lambda})$ is said to be an efficient solution of $(\mathrm{MSID_{MW}})$, if there does not exist any other $(u, \alpha, \lambda) \in F_{MW}$ satisfying

$$\widetilde{f}(\tilde{u}) \preceq \widetilde{f}(u).$$

Definition 4.2. Let $(\tilde{u}, \tilde{\alpha}, \tilde{\lambda}) \in F_{MW}$. Then, $(\tilde{u}, \tilde{\alpha}, \tilde{\lambda})$ is said to be a weakly efficient solution of (MSID$_{\mathrm{MW}}$), if there does not exist any other $(u, \alpha, \lambda) \in F_{MW}$ satisfying

$$\widetilde{f}(\tilde{u}) \prec \widetilde{f}(u).$$

The following example illustrates the concept of efficient solution of Mond-Weir dual problem.

Example 4.3. Let us consider the Poincaré half-plane defined as follows

$$\mathscr{H} := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}.$$

Let us consider the following geodesic convex set on the Hadamard manifold $\mathscr{H}$ as follows:

$$S := \{x \in \mathscr{H} : x_1 > -\frac{1}{2}\}.$$

Let us consider the multiobjective semi-infinite problem (P_1) as defined in Example 3.3. We denote the feasible set of (P_1) by F.
The Mond-Weir dual problem related to (P_1), denoted by (D_{M1}), may be formulated as follows:

$$\begin{aligned}
(D_{M1}) \quad \text{Maximize } \widetilde{f}(u) := \Big(f_1(u), f_2(u)\Big) &:= \left(\frac{u_1^2}{2u_2} + \frac{u_2}{2}, \frac{\ln^2 \frac{u_1 + \frac{1}{2}}{u_2}}{2}\right) \\
\text{subject to } \sum_{i \in I} \alpha_i \operatorname{grad} f_i(u) &+ \sum_{j \in J} \lambda_j \operatorname{grad} g_j(u) = 0, \\
&\sum_{j \in J} \lambda_j g_j(u) \geq 0, \\
&u \in S, \alpha \in \mathbb{R}^m_+ \setminus \{0\}, \lambda \in \mathbb{R}^{|J|}_+.
\end{aligned}$$

The feasible set of (D_{M1}), denoted by F_D, is given by

$$\begin{aligned}
F_D := \{(u, \alpha, \lambda) \in & S \times \mathbb{R}^m_+ \times \mathbb{R}^{|J|}_+ : \alpha \neq 0, \\
& \sum_{i \in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(u) = 0, \sum_{j \in J} \lambda_j g_j(u) \geq 0\}.
\end{aligned}$$

Let us consider the point $\bar{x} = (0, \frac{1}{2}) \in F$, which is an efficient solution of (P$_1$). Moreover, we claim that there exists $\bar{\alpha} \in \mathbb{R}^2$, satisfying $\sum_{i=1}^{2} \bar{\alpha}_i = 1$ and $\bar{\lambda} \in \mathscr{A}(\bar{x})$, such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_D$.
Let $\bar{\lambda} : J \to \mathbb{R}$ be defined as follows

$$\bar{\lambda}_j = \begin{cases} \frac{1}{16}, & j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we see that

$$\bar{\lambda}_j g_j(\bar{x}) = 0, \quad \forall j \in J.$$

Hence, there exist $\bar{\alpha} = (\bar{\alpha_1}, \bar{\alpha_2}) = \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^2$, satisfying $\sum_{i=1}^{2} \bar{\alpha}_i = 1$ and $\bar{\lambda} \in \mathscr{A}(\bar{x})$, such that

$$\sum_{i=1}^{2} \bar{\alpha}_i \operatorname{grad} f_i(\bar{x}) + \sum_{j \in J} \bar{\lambda}_j \operatorname{grad} g_j(\bar{x}) = \frac{1}{2}(0, \frac{1}{8}) + \frac{1}{2}(0,0) + \frac{1}{16}(0,-1) = (0,0).$$

That is, $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a feasible point of the Mond-Weir dual problem D_{M1}. We can verify that there does not exist any other $(u, \alpha, \lambda) \in F_D$ satisfying

$$\widetilde{f}(\bar{x}) \prec \widetilde{f}(u).$$

Thus, $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_D$ is a weakly efficient solution of the Mond-Weir dual problem.

The following theorem establishes weak duality relating (MSIP) and (MSID_{MW}).

Theorem 4.4 (Weak duality)**.** *Let $x \in F$ and $(u, \alpha, \lambda) \in F_{MW}$. Then the following statements are true.*

(i) If f_i is geodesic pseudoconvex at u for all $i \in I$, and $\sum_{j \in J} \lambda_j g_j$ is geodesic quasiconvex at u, then

$$f(x) \nprec \widetilde{f}(u).$$

(ii) If f_i is geodesic strictly pseudoconvex at u for all $i \in I$, $\sum_{j \in J} \lambda_j g_j$ is geodesic quasiconvex at u, then

$$f(x) \npreceq \widetilde{f}(u).$$

Proof. Since $x \in F$, we have

$$g_j(x) \leq 0, \quad \forall j \in J. \tag{13}$$

Also, as $(u, \alpha, \lambda) \in F_{MW}$, it follows that

$$\sum_{i \in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(u) = 0, \tag{14}$$

$$\sum_{j \in J} \lambda_j g_j(u) \geq 0. \tag{15}$$

(i) On the contrary, let us assume that

$$f(x) \prec \widetilde{f}(u, \alpha, \lambda) = (f_1(u), f_2(u), ..., f_m(u)). \tag{16}$$

Then, it follows from (16) that

$$f_i(x) < f_i(u), \quad \forall i \in I. \tag{17}$$

Since $f_i(x)$ is geodesic pseudoconvex for all $i \in I$, inequality (17) implies that

$$\left\langle \operatorname{grad} f_i(u), \exp{}_u^{-1}(x) \right\rangle_u < 0, \quad \forall i \in I. \tag{18}$$

Since $\alpha \in \mathbb{R}_+^m \setminus \{0\}$, from (18) we obtain

$$\left\langle \sum_{i\in I} \alpha_i \operatorname{grad} f_i(u), \exp{}_u^{-1}(x) \right\rangle_u < 0. \tag{19}$$

From inequality (13) and $\lambda \in \mathbb{R}_+^{|J|}$, we have

$$\sum_{j\in J} \lambda_j g_j(x) \leq 0. \tag{20}$$

Combining inequality (20) with (15), we get

$$\sum_{j\in J} \lambda_j g_j(x) \leq 0 \leq \sum_{j\in J} \lambda_j g_j(u).$$

Then, from the quasiconvexity of $\sum_{j\in J} \lambda_j g_j$ at u, it follows that

$$\left\langle \sum_{j\in J} \lambda_j \operatorname{grad} g_j(u), \exp{}_u^{-1}(x) \right\rangle_u \leq 0. \tag{21}$$

Adding (19) and (21), we get

$$\left\langle \sum_{i\in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j\in J} \lambda_j \operatorname{grad} g_j(u), \exp{}_u^{-1}(x) \right\rangle_u < 0,$$

which is a contradiction to (14). This proves that $f(x) \not< \widetilde{f}(u)$.
(ii) On the contrary, let us assume that

$$f(x) \preceq \widetilde{f}(u, \alpha, \lambda) = f(u).$$

This implies that

$$\begin{aligned} &f_i(x) \leq f_i(u), \quad \forall i \in I,\ i \neq p, \\ &f_p(x) < f_p(u), \quad \text{for some } p \in I. \end{aligned} \tag{22}$$

It follows that $x \neq u$. Since $f_i(x)$ is geodesic strictly pseudoconvex at u for all $i \in I$, from (22), we have

$$\left\langle \operatorname{grad} f_i(u), \exp_u^{-1}(x) \right\rangle_u < 0, \quad \forall i \in I. \tag{23}$$

Since $\alpha \in \mathbb{R}^m_+ \setminus \{0\}$, then from (23), we obtain

$$\left\langle \sum_{i \in I} \alpha_i \operatorname{grad} f_i(u), \exp_u^{-1}(x) \right\rangle_u < 0. \tag{24}$$

From inequality (13) and $\lambda \in \mathbb{R}^{|J|}_+$, we have

$$\sum_{j \in J} \lambda_j g_j(x) \leq 0.$$

Combining this with (15), we get

$$\sum_{j \in J} \lambda_j g_j(x) \leq 0 \leq \sum_{j \in J} \lambda_j g_j(u).$$

Then, from the quasiconvexity of $\sum_{j \in J} \lambda_j g_j$ at u, it follows that,

$$\left\langle \sum_{j \in J} \lambda_j \operatorname{grad} g_j(u), \exp_u^{-1}(x) \right\rangle_u \leq 0. \tag{25}$$

Adding (24) and (25), we get

$$\left\langle \sum_{i \in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(u), \exp_u^{-1}(x) \right\rangle_u < 0,$$

which is a contradiction to (14). This proves that $f(x) \not\prec \widetilde{f}(u)$. □

The following theorem establishes strong duality relating (MSIP) and (MSID_{MW}).

Theorem 4.5 (Strong duality)**.** *Let $\bar{x}$ be a weakly efficient solution of (MSIP) such that Abadie constraint qualification (ACQ) is satisfied at $\bar{x}$. Then, there exist $\bar{\alpha} \in \mathbb{R}^m_+ \setminus \{0\}$ and $\bar{\lambda} \in \mathscr{A}(\bar{x})$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_{MW}$ and*

$$f(\bar{x}) = \widetilde{f}(\bar{x}).$$

Moreover, the following statements are true.

(i) If the assumptions of weak duality (Theorem 4.4 (i)) hold true, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a weakly efficient solution of ($MSID_{MW}$).

(ii) If the assumptions of weak duality (Theorem 4.4 (ii)) hold true, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an efficient solution of (MSID$_{MW}$).

Proof. Since, $\bar{x}$ is a weakly efficient solution of (MSIP) and (ACQ) is satisfied at $\bar{x}$, we infer from Theorem 3.1, that there exist $\bar{\alpha} \in \mathbb{R}^m_+ \setminus \{0\}$ satisfying $\sum_{i \in I} \bar{\alpha}_i = 1$ and $\bar{\lambda} \in \mathscr{A}(\bar{x})$ such that

$$\sum_{i \in I} \bar{\alpha}_i \operatorname{grad} f_i(\bar{x}) + \sum_{j \in J} \bar{\lambda}_j \operatorname{grad} g_j(\bar{x}) = 0. \tag{26}$$

Since $\bar{\lambda} \in \mathscr{A}(\bar{x})$, we have

$$\bar{\lambda}_j g_j(\bar{x}) = 0, \quad \forall j \in J,$$

and hence

$$\sum_{j \in J} \bar{\lambda}_j g_j(\bar{x}) = 0. \tag{27}$$

Equations (26) and (27) implies that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_{MW}$. Thus, we have $f(\bar{x}) = \widetilde{f}(\bar{x})$.

(i) From weak duality theorem (Theorem 4.4 (i)), it follows that for any $(u, \alpha, \lambda) \in F_{MW}$, we have

$$\widetilde{f}(\bar{x}) \not< \widetilde{f}(u).$$

This proves that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a weakly efficient solution of (MSID$_{\mathrm{MW}}$).

(ii) From weak duality theorem (Theorem 4.4 (ii)), it follows that for any $(u, \alpha, \lambda) \in F_{MW}$, we have

$$\widetilde{f}(\bar{x}) \not\leq \widetilde{f}(u).$$

This proves that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an efficient solution of (MSID$_{\mathrm{MW}}$).

□

The following example illustrates strong duality theorem relating (MSIP) and (MSID$_{\mathrm{MW}}$).

Example 4.6. Let us consider the Poincaré half-plane defined as follows

$$\mathscr{H} := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}.$$

Let us consider the following geodesic convex set on the Hadamard manifold $\mathscr{H}$ as follows:

$$S := \{x \in \mathscr{H} : x_1 > -\frac{1}{2}\}.$$

Let us consider the multiobjective semi-infinite problem (P_1) as defined in Example 3.3. We denote the feasible set of (P_1) by F.

The Mond-Weir dual problem related to (P_1), denoted by (D_{M1}), may be formulated as follows:

$$(D_{M1})\quad \text{Maximize } \widetilde{f}(u) := \Big(f_1(u), f_2(u)\Big) := \left(\frac{u_1^2}{2u_2} + \frac{u_2}{2}, \frac{\ln^2 \frac{u_1+\frac{1}{2}}{u_2}}{2}\right)$$
$$\text{subject to } \sum_{i\in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j\in J} \lambda_j \operatorname{grad} g_j(u) = 0,$$
$$\sum_{j\in J} \lambda_j g_j(u) \geq 0,$$
$$u \in S, \alpha \in \mathbb{R}^m_+ \setminus \{0\}, \lambda \in \mathbb{R}^{|J|}_+.$$

The feasible set of (D_{M1}), denoted by F_D, is given by

$$F_D := \{(u, \alpha, \lambda) \in S \times \mathbb{R}^m_+ \times \mathbb{R}^{|J|}_+ : \alpha \neq 0, \sum_{i\in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j\in J} \lambda_j \operatorname{grad} g_j(u) = 0, \sum_{j\in J} \lambda_j g_j(u) \geq 0\}.$$

Let us consider the point $\bar{x} = (0, \frac{1}{2}) \in F$. Then, it can be verified that

$$\mathscr{T}(\bar{x}, F) = \{v = (v_1, v_2) \in T_{\bar{x}}S : v_1, v_2 \geq 0\}.$$

Also, we have the following

$$\operatorname{grad} f_1(\bar{x}) = \left(0, \frac{1}{8}\right), \quad \operatorname{grad} f_2(\bar{x}) = (0, 0), \quad \operatorname{grad} g_j(\bar{x}) = (0, -j).$$

Then, it follows from Example 3.3 that (ACQ) is satisfied at the feasible point $\bar{x} = (0, \frac{1}{2})$. We can check that $\bar{x}$ is an efficient solution of (P_1). Thus, we see that all the assumptions for strong duality of Mond-Weir dual problem (Theorem 4.5) are satisfied.
Let $\lambda : J \to \mathbb{R}$ be defined as follows

$$\lambda_j = \begin{cases} \frac{1}{16}, & j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we see that

$$\lambda_j g_j(\bar{x}) = 0, \quad \forall j \in J.$$

Hence, there exist $\alpha = (\alpha_1, \alpha_2) = \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathbb{R}^2$, satisfying $\sum_{i=1}^{2} \alpha_i = 1$ and $\lambda \in \mathscr{A}(\bar{x})$, such that

$$\sum_{i=1}^{2} \alpha_i \operatorname{grad} f_i(\bar{x}) + \sum_{j\in J} \lambda_j \operatorname{grad} g_j(\bar{x}) = \frac{1}{2}(0, \frac{1}{8}) + \frac{1}{2}(0, 0) + \frac{1}{16}(0, -1) = (0, 0).$$

That is, $(\bar{x}, \alpha, \lambda)$ is a feasible point of the Mond-Weir dual problem D_{M1}. Further, we see that,

$$f(\bar{x}) = \widetilde{f}(\bar{x}).$$

Now, from Example 3.3, we see that f_i is geodesic pseudoconvex for all $i \in I$. Further, it can be verified that $\sum_{j\in J} \lambda_j g_j$ is geodesic quasiconvex. Thus from the strong duality theorem, it can be verified that $(\bar{x}, \alpha, \lambda)$ is a weakly efficient solution of D_{M1}.

The following theorem establishes the strict converse duality relating (MSIP) and (MSID_{MW}).

Theorem 4.7 (Strict converse duality)**.** *Let x^* be a weakly efficient solution of (MSIP) such that Abadie constraint qualification (ACQ) is satisfied at x^*. Let $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ be a weakly efficient solution of ($MSID_{MW}$). If f_i is geodesic strictly pseudoconvex at $\bar{x}$ for all $i \in I$ and $\sum\limits_{j\in J} \lambda_j g_j$ is geodesic quasiconvex at $\bar{x}$, then $x^* = \bar{x}$.*

Proof. On the contrary, let us assume that $x^* \neq \bar{x}$. Since x^* is a weakly efficient solution of (MSIP) and (ACQ) is satisfied at x^*, we can infer from Theorem 4.5 that there exist $\alpha^* \in \mathbb{R}^m_+ \setminus \{0\}$ and $\lambda^* \in \mathscr{A}(x^*)$ such that $(x^*, \alpha^*, \lambda^*) \in F_{MW}$ and

$$f(x^*) = \widetilde{f}(x^*).$$

Further, it also follows from Theorem 4.5 that $(x^*, \alpha^*, \lambda^*)$ is an efficient solution of (MSID_{MW}). Since $x^* \in F$ and $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_{MW}$, then from Theorem 4.4 (ii), it follows that

$$\widetilde{f}(\bar{x}) \prec f(x^*) = \widetilde{f}(x^*),$$

which is a contradiction. This completes the proof. □

Now, we give an example to illustrate the results obtained for Mond-Weir duality.

Example 4.8. Let us consider the set $\mathscr{H} \subset \mathbb{R}^2$ as follows

$$\mathscr{H} := \{x = (x_1, x_2) \in \mathbb{R}^2, x_1, x_2 > 0\}.$$

Then $\mathscr{H}$ is a Riemannian manifold (see for instance, [7, 46, 56], and Example 4.4 of [44]). $\mathscr{H}$ is equipped with the metric as defined below

$$\langle u, v\rangle_x = \langle \mathscr{G}(x)u, v\rangle, \quad \forall u, v \in T_x\mathscr{H} = \mathbb{R}^2,$$

where $\langle \cdot, \cdot\rangle$ is the standard inner product on $\mathbb{R}^2$ and

$$\mathscr{G}(x) = \begin{pmatrix} \frac{1}{x_1^2} & 0 \\ 0 & \frac{1}{x_2^2} \end{pmatrix}.$$

Since the sectional curvature of $\mathscr{H}$ is 0, which is non positive, $\mathscr{H}$ is also a Hadamard manifold. Also, $\mathscr{H}$ is a geodesic convex set. The second Christoffel symbols are as follows:

$$\Gamma^{x_1} = \begin{pmatrix} -\frac{1}{x_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \Gamma^{x_2} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{x_2} \end{pmatrix}.$$

The Riemannian distance between $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathscr{H}$ is given by

$$d(x, y) = \left\| \ln \frac{x_1}{y_1}, \ln \frac{x_2}{y_2} \right\|.$$

The exponential map $\exp_x : T_x\mathscr{H} \to \mathscr{H}$ for any $u \in T_x\mathscr{H}$ is given by

$$\exp_x(u) = (x_1 e^{\frac{u_1}{x_1}}, x_2 e^{\frac{u_2}{x_2}}), \quad \forall u = (u_1, u_2) \in \mathscr{H}.$$

The inverse of the exponential map $\exp_x^{-1} : \mathscr{H} \to T_x\mathscr{H}$ for any $y \in \mathscr{H}$ is given by

$$\exp_x^{-1}(y) = \left(x_1 \ln \frac{y_1}{x_1}, x_2 \ln \frac{y_2}{x_2} \right).$$

We consider the following semi-infinite programming problem on $\mathscr{H}$

$$\begin{aligned} (P_2) \quad & \text{Minimize } f(x) = 2\sqrt{x_1} + \sqrt{x_2}, \\ & \text{subject to } \ g_j(x) = \frac{1}{2} - \frac{1-j}{2} \ln x_1 - \frac{j}{2} \ln x_2 \le 0, \quad j \in J = [0,1]. \end{aligned}$$

Here, $f, g_j : \mathscr{H} \to \mathbb{R}^2$. The feasible set F for the problem is

$$F = \{x \in \mathscr{H}, x_1 \ge e, x_2 \ge e\}.$$

The Mond-Weir dual problem related to (P_2) may be formulated as

$$\begin{aligned} (\mathrm{P_{MW}}) \quad & \text{Maximize } \widetilde{f}(u) = f(u) = 2\sqrt{u_1} + \sqrt{u_2}, \\ & \text{subject to} \quad \alpha \text{ grad } f(u) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(u) = (0,0), \quad \sum_{j \in J} \lambda_j g_j(u) \ge 0, \\ & \qquad u \in \mathscr{H}, \alpha \in \mathbb{R}_+ \setminus \{0\}, \lambda \in \mathbb{R}_+^{|J|}. \end{aligned}$$

The feasible set of $(\mathrm{P_{MW}})$ is given by

$$\begin{aligned} F_{\mathrm{MW}} = \{ & (u, \alpha, \lambda) \in \mathscr{H} \times \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}_+^{|J|}, \\ & \alpha \text{ grad } f(u) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(u) = (0,0), \sum_{j \in J} \lambda_j g_j(u) \ge 0 \}. \end{aligned}$$

Let us consider the feasible point $\bar{x} = (e, e) \in F_{\mathrm{MW}}$. Since

$$g_j(e,e) = \frac{1}{2} - \frac{1-j}{2} - \frac{j}{2} = 0, \quad \forall j \in J,$$

we have $L(\bar{x}) = J$. Let v be any arbitrary element in the contingent cone $\mathscr{T}(F, \bar{x})$. Then, there exist $t_k \downarrow 0$ and $v_k \in T_{\bar{x}}\mathscr{H} = \mathbb{R}^2$ such that $v^k = \left(v_1^k, v_2^k\right) \to v = (v_1, v_2)$. Also, we have

$$\exp_{\bar{x}}(t_k v^k) = \left(e.e^{\frac{t_k v_1^k}{e}}, e.e^{\frac{t_k v_2^k}{e}}\right) \in F, \quad \forall k.$$

This gives us

$$e.e^{\frac{t_k v_1^k}{e}} \geq e \text{ and } e.e^{\frac{t_k v_2^k}{e}} \geq e, \quad \forall k,$$

which implies

$$\frac{t_k v_1^k}{e} \geq 0 \text{ and } \frac{t_k v_2^k}{e} \geq 0, \quad \forall k, \text{ or equivalently, } v_1^k \geq 0 \text{ and } v_2^k \geq 0, \quad \forall k.$$

Letting k to infinity, we can conclude that

$$v_1 \geq 0, \quad v_2 \geq 0.$$

Hence, if follows that $\mathscr{T}(F, \bar{x}) \subseteq \mathbb{R}^2_+$. Similarly, it can be proved that $\mathbb{R}^2_+ \subseteq \mathscr{T}(F, \bar{x})$. Thus, we have

$$\mathscr{T}(F, \bar{x}) = \mathbb{R}^2_+.$$

Also, we have the following

$$\operatorname{grad} f(x) = \mathscr{G}(x)^{-1} \begin{pmatrix} \frac{1}{\sqrt{x_1}} \\ \frac{1}{2\sqrt{x_2}} \end{pmatrix} = \begin{pmatrix} x_1\sqrt{x_1} \\ \frac{x_2\sqrt{x_2}}{2} \end{pmatrix},$$

$$\operatorname{grad} g_j(x) = \mathscr{G}(x)^{-1} \begin{pmatrix} -\frac{1-j}{2x_1} \\ -\frac{j}{2x_2} \end{pmatrix} = \left(-\frac{1-j}{2}x_1, -\frac{j}{2}x_2\right), \quad \forall j \in J.$$

Substituting $\bar{x} = (e, e)$ for $x = (x_1, x_2)$ in the above equations, we get

$$\operatorname{grad} f(\bar{x}) = \left(e\sqrt{e}, \frac{e}{2}\sqrt{e}\right),$$

$$\operatorname{grad} g_j(\bar{x}) = \left(-\frac{1-j}{2}e, -\frac{j}{2}e\right), \quad \forall j \in J.$$

Hence, we obtain the following

$$\bigcup_{j \in L(\bar{x})} \operatorname{grad} g_\ell(\bar{x}) = \left\{x \in T_{\bar{x}}\mathscr{H} : x_1 + x_2 = -\frac{e}{2},\ x_1 \leq 0, x_2 \leq 0\right\},$$

and

$$\left(\bigcup_{j\in L(\bar{x})} \operatorname{grad} g_j(\bar{x})\right)^{-} = \{x^* \in T_{\bar{x}}\mathscr{H} : \langle x^*, y\rangle \leq 0, \forall y \in \bigcup_{j\in L(\bar{x})} \operatorname{grad} g_j(\bar{x})\}$$
$$= \{x^* \in T_{\bar{x}}\mathscr{H} : x_1^* \geq 0, x_2^* \geq 0\} \subseteq \mathscr{T}(F_{MW}, \bar{x}).$$

Also, the positive conic hull

$$\operatorname{pos} \bigcup_{\ell\in L(\bar{x})} \operatorname{grad} g_\ell(\bar{x}) = \{x \in T_{\bar{x}}\mathscr{H} \mid x_1 \leq 0, x_2 \leq 0\}$$

is closed. This implies that Abadie constraint qualification (ACQ) holds at $\bar{x}$.
We can check that $\bar{x}$ is an efficient solution of (P_2). Thus, we see that all the assumptions for strong duality of Mond-Weir dual problem (Theorem 4.5) are satisfied.
Let $\bar{\lambda} : J \to \mathbb{R}$ be defined as follows:

$$\bar{\lambda}_j = \begin{cases} 3\sqrt{e}, & \text{if } j = \frac{1}{3}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we see that

$$\bar{\lambda}_j g_j(\bar{x}) = 0, \quad \forall j \in J.$$

This implies that $\bar{\lambda} \in \mathscr{A}(\bar{x})$. Then, there exist $\bar{\alpha} = 1 \in \mathbb{R}$ and $\bar{\lambda} \in \mathscr{A}(\bar{x})$, such that

$$\bar{\alpha} \operatorname{grad} f(\bar{x}) + \sum_{j\in J} \bar{\lambda}_j \operatorname{grad} g_j(\bar{x}) = \left(e\sqrt{e}, \frac{e}{2}\sqrt{e}\right) + 3\sqrt{e}\left(-\frac{1-\frac{1}{3}}{2}e, -\frac{\frac{1}{3}}{2}e\right)$$
$$= \left(e\sqrt{e}, \frac{e}{2}\sqrt{e}\right) + 3\sqrt{e}\left(-\frac{1}{3}e, -\frac{1}{6}e\right)$$
$$= (0, 0).$$

This shows that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_{MW}$. Thus, we have

$$f(\bar{x}) = \widetilde{f}(\bar{x}).$$

Now, we observe that

$$f(x) = 2\sqrt{x_1} + \sqrt{x_2} = \frac{4\sqrt{x_1} + 2\sqrt{x_2}}{2} = \frac{f_1(x)}{2},$$
$$g_j(x) = \frac{1}{2} - \frac{1-j}{2}\ln x_1 - \frac{j}{2}\ln x_2 = \frac{1 - (1-j)\ln x_1 - j\ln x_2}{2} = \frac{g_j'(x)}{2},$$

where,

$$f_1(x) = 4\sqrt{x_1} + 2\sqrt{x_2},$$
$$g_j'(x) = 1 - (i-j)\ln x_1 - j\ln x_2.$$

It can be verified in the similar lines of Example 3.3 that the hyperbolic Hessian (or, the second covariant derivative) $H^g f_1(x)$ is a positive semidefinite matrix. Thus, $f(x)$ is a ratio of a geodesic convex function and a positive affine function. Then, from Theorem 2.8, it follows that f is a geodesic pseudoconvex function. Also, $H^g\left(\sum_{j\in J}\lambda_j g_j'(x)\right)$ is positive semidefinite matrix. This implies that $\sum_{j\in J}\lambda_j g_j$ is geodesic quasiconvex. Thus, we see that all the assumptions in Theorem 4.5 are satisfied. It can be verified that $(\bar{x},\bar{\alpha},\bar{\lambda})$ is a weakly efficient solution of the Mond-Weir dual problem ($\mathrm{P_{MW}}$).

4.2. Wolfe Duality

Let us consider that $u \in S \subset \mathscr{H}$, where S is an open geodesic convex set in $\mathscr{H}$, $\alpha \in \mathbb{R}^m_+$, with $\sum_{i\in I}\alpha_i = 1$ and $\lambda \in \mathbb{R}^{|J|}_+$. Then, the Wolfe dual problem of (MSIP), denoted by ($\mathrm{MSID_W}$), is formulated as follows:

$$\begin{aligned}(\mathrm{MSID_W})\quad &\text{Maximize } \mathscr{L}(u,\alpha,\lambda) := f(u) + \sum_{j\in J}\lambda_j g_j(u)e,\\ &\text{subject to } \sum_{i\in I}\alpha_i \operatorname{grad} f_i(u) + \sum_{j\in J}\lambda_j \operatorname{grad} g_j(u) = 0,\end{aligned}$$

where $u \in S \subset \mathscr{H}$, $\alpha \in \mathbb{R}^m_+$, with $\sum_{i\in I}\alpha_i = 1$, $\lambda \in \mathbb{R}^{|J|}_+$, and $e = (1,1,...,1)$. The feasible set of ($\mathrm{MSID_W}$), denoted by ($\mathrm{F_W}$), is given by

$$\begin{aligned}F_W = \{(u,\alpha,\lambda) \in &S \times \mathbb{R}^m_+ \times \mathbb{R}^{|J|}_+ : \sum_{i\in I}\alpha_i = 1, \text{ and}\\ &\sum_{i\in I}\alpha_i \operatorname{grad} f_i(u) + \sum_{j\in J}\lambda_j \operatorname{grad} g_j(u) = 0\}.\end{aligned}$$

We define $h : \mathscr{H} \to \mathbb{R}$ as follows:

$$h(x) := \left(\sum_{i\in I}\alpha_i f_i + \sum_{j\in J}\lambda_j g_j\right)(x) = \sum_{i\in I}\alpha_i f_i(x) + \sum_{j\in J}\lambda_j g_j(x).$$

The following definitions of efficient solution and weakly efficient solution of the Wolfe dual problem ($\mathrm{MSID_W}$) are from Tung and Tam [56].

Definition 4.9. Let $(\bar{u},\bar{\alpha},\bar{\lambda}) \in F_W$. Then $(\bar{u},\bar{\alpha},\bar{\lambda})$ is said to be an efficient solution of ($\mathrm{MSID_W}$) if there does not exist any other $(u,\alpha,\lambda) \in F_W$, such that

$$\mathscr{L}(\bar{u},\bar{\alpha},\bar{\lambda}) \preceq \mathscr{L}(u,\alpha,\lambda).$$

Definition 4.10. Let $(\bar{u}, \bar{\alpha}, \bar{\lambda}) \in F_W$. Then $(\bar{u}, \bar{\alpha}, \bar{\lambda})$ is said to be a weakly efficient solution of (MSID$_\mathrm{W}$), if there does not exist any other $(u, \alpha, \lambda) \in F_W$, such that

$$\mathscr{L}(\bar{u}, \bar{\alpha}, \bar{\lambda}) \prec \mathscr{L}(u, \alpha, \lambda).$$

The following theorem establishes weak duality relating (MSIP) and (MSID$_\mathrm{W}$).

Theorem 4.11 (Weak duality)**.** *Let $x \in F$ and $(u, \alpha, \lambda) \in F_W$. Then the following statements are true.*

(i) If h is geodesic pseudoconvex at u, then $f(x) \not\prec \mathscr{L}(u, \alpha, \lambda)$.

(ii) If h is geodesic strictly pseudoconvex at u, then $f(x) \not\preceq \mathscr{L}(u, \alpha, \lambda)$.

Proof. Since $x \in F$, we have

$$g_j(x) \leq 0, \quad \forall j \in J. \tag{28}$$

Also, as $(u, \alpha, \lambda) \in F_W$, it follows that

$$\sum_{i \in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(u) = 0. \tag{29}$$

(i) On the contrary, let us assume that,

$$f(x) \prec \mathscr{L}(u, \alpha, \lambda) = f(u) + \sum_{j \in J} \lambda_j g_j(u) e.$$

Then, it follows that

$$f_i(x) < f_i(u) + \sum_{j \in J} \lambda_j g_j(u), \quad \forall i \in I. \tag{30}$$

Since $\alpha \in \mathbb{R}^m_+$ with $\sum\limits_{i \in I} \alpha_i = 1$, $\lambda \in \mathbb{R}^{|J|}_+$, we have the following:

$$\begin{aligned}
\sum_{i \in I} \alpha_i f_i(x) + \sum_{j \in J} \lambda_j g_j(x) &\leq \sum_{i \in I} \alpha_i f_i(x) \\
&< \sum_{i \in I} \alpha_i \left(f_i(u) + \sum_{j \in J} \lambda_j g_j(u) \right) \\
&= \sum_{i \in I} \alpha_i f_i(u) + \sum_{i \in I} \alpha_i \sum_{j \in J} \lambda_j g_j(u) \\
&= \sum_{i \in I} \alpha_i f_i(u) + \sum_{j \in J} \lambda_j g_j(u).
\end{aligned}$$

Thus, it follows that

$$\left(\sum_{i \in I} \alpha_i f_i + \sum_{j \in J} \lambda_j g_j \right)(x) < \left(\sum_{i \in I} \alpha_i f_i + \sum_{j \in J} \lambda_j g_j \right)(u),$$

that is,

$$h(x) < h(u).$$

Since h is a geodesic pseudoconvex function at u, we have

$$h(x) < h(u) \implies \left\langle \operatorname{grad} h(u), \exp_u^{-1}(x) \right\rangle_u < 0.$$

This implies that

$$\left\langle \sum_{i\in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j\in J} \lambda_j \operatorname{grad} g_j(u), \exp_u^{-1}(x) \right\rangle_u < 0,$$

which is a contradiction to (29). This proves that

$$f(x) \not< \mathscr{L}(u, \alpha, \lambda).$$

(ii) On the contrary, let us assume that $f(x) \not\leq \mathscr{L}(u, \alpha, \lambda) = f(u) + \sum_{j\in J} \lambda_j g_j(u)e$. This implies that

$$\begin{aligned} f_i(x) &\leq f_i(u) + \sum_{j\in J} \lambda_j g_j(u), \quad \forall i \in I, \ i \neq p, \\ f_p(x) &< f_p(u) + \sum_{j\in J} \lambda_j g_j(u), \quad \text{for atleast one } p \in I. \end{aligned}$$

It follows that $x \neq u$. Since $\alpha \in \mathbb{R}^m_+$ with $\sum_{i\in I} \alpha_i = 1$, $\lambda \in \mathbb{R}^{|J|}_+$, we have the following:

$$\begin{aligned} \sum_{i\in I} \alpha_i f_i(x) + \sum_{j\in J} \lambda_j g_j(x) &\leq \sum_{i\in I} \alpha_i f_i(x) \\ &< \sum_{i\in I} \alpha_i \Big(f_i(u) + \sum_{j\in J} \lambda_j g_j(u) \Big) \\ &= \sum_{i\in I} \alpha_i f_i(u) + \sum_{i\in I} \alpha_i \sum_{j\in J} \lambda_j g_j(u) \\ &= \sum_{i\in I} \alpha_i f_i(u) + \sum_{j\in J} \lambda_j g_j(u). \end{aligned}$$

Hence, we have

$$\Big(\sum_{i\in I} \alpha_i f_i + \sum_{j\in J} \lambda_j g_j \Big)(x) < \Big(\sum_{i\in I} \alpha_i f_i + \sum_{j\in J} \lambda_j g_j \Big)(u),$$

that is,

$$h(x) < h(u).$$

Since h is a geodesic strictly pseudoconvex at u, we have

$$h(x) < h(u) \implies \left\langle \operatorname{grad} h(u), \exp_u^{-1}(x) \right\rangle_u < 0.$$

Thus, we obtain

$$\left\langle \sum_{i\in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j\in J} \lambda_j \operatorname{grad} g_j(u), \exp_u^{-1}(x) \right\rangle_u < 0,$$

which is a contradiction to (29). This completes the proof. □

Example 4.12. Let us consider the Poincaré half-plane defined as follows

$$\mathscr{H} := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}.$$

Let us consider the following geodesic convex set on the Hadamard manifold $\mathscr{H}$ as follows:

$$S := \{x \in \mathscr{H} : x_1 > -\frac{1}{2}\}.$$

Let us consider the multiobjective semi-infinite programming problem P_1 as defined in Example 3.3. The feasible set on the problem P_1 is denoted by F.
The Wolfe dual problem related to (P_1), denoted by (D_1), may be formulated as follows

$$(\mathrm{D}_1) \quad \text{Maximize } \mathscr{L}(u, \alpha, \lambda) := f(u) + \sum_{j\in J} \lambda_j g_j(u) e,$$

$$\text{subject to } \sum_{i\in I} \alpha_i \operatorname{grad} f_i(u) + \sum_{j\in J} \lambda_j \operatorname{grad} g_j(u) = 0,$$

The feasible set of D_1 is denoted by F_1. Let us consider the point $\bar{x} = (0, \frac{1}{2}) \in F$. Then, it can be verified that

$$\mathscr{T}(\bar{x}, F) = \{v = (v_1, v_2) \in T_{\bar{x}} S : v_1, v_2 \geq 0\}.$$

Also, we have the following

$$\operatorname{grad} f_1(\bar{x}) = \frac{1}{4}\left(0, \frac{1}{2}\right) = \left(0, \frac{1}{8}\right), \quad \operatorname{grad} f_2(\bar{x}) = (0, 0), \quad \operatorname{grad} g_j(\bar{x}) = (0, -j).$$

Then, it follows from Example 3.3 that (ACQ) is satisfied at the feasible point $\bar{x} = (0, \frac{1}{2})$.

Let $\lambda : J \to \mathbb{R}$ be defined as follows

$$\lambda(j) = \begin{cases} \frac{1}{8}, & j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, there exist $\alpha = (\alpha_1, \alpha_2) = (1, 0) \in \mathbb{R}^2$, satisfying $\sum\limits_{i=1}^{2} \alpha_i = 1$ and $\lambda \in \mathscr{A}(\bar{x})$, such that

$$\sum_{i=1}^{2} \alpha_i \operatorname{grad} f_i(\bar{x}) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(\bar{x}) = 1(0, \frac{1}{8}) + 0(0,0) + \frac{1}{8}(0, -1) = 0.$$

Thus, $(\bar{x}, \alpha, \lambda) \in F_1$. Then, the function h is defined as:

$$\begin{aligned} h :&= \alpha_1 f_1 + \alpha_2 f_2 + \lambda_1 g_1 \\ &= f_1 + \frac{1}{8} g_1 \\ &= \frac{4\frac{x_1^2}{x_2} + 4x_2 + \frac{1}{x_2} - 16}{8} \quad = \frac{H}{8}, \text{say,} \end{aligned}$$

where, $H = 4\frac{x_1^2}{x_2} + 4x_2 + \frac{1}{x_2} - 16$. Then, we obtain the following:

$$\nabla H = \begin{pmatrix} \frac{8x_1}{x_2} \\ \frac{-4x_1^2}{x_2^2} + 4 - \frac{1}{x_2^2} \end{pmatrix}.$$

Then, it follows that

$$\nabla^2 H = \begin{pmatrix} \frac{8}{x_2} & \frac{-8x_1}{x_2^2} \\ \frac{-8x_1}{x_2^2} & \frac{8x_1^2}{x_2^3} + \frac{2}{x^3} \end{pmatrix}.$$

Now, we see that, the (hyperbolic) Hessian, or the second-order covariant derivative, of H is given by

$$\begin{aligned} H^g H &= \nabla^2 H - \nabla H \Gamma \\ &= \begin{bmatrix} \frac{4x_1^2 + 4x_2^2 + 1}{x_2^3} & 0 \\ 0 & \frac{4x_1^2 + 6x_2^2 + 1}{x_2^3} \end{bmatrix}. \end{aligned}$$

is a positive semidefinite matrix as all its eigen values are non negative. Thus, H is geodesic convex at $(0, \frac{1}{2})$. Then, h is a ratio of a geodesic convex function and a positive affine function. Then, from Theorem 2.8, it follows that h is a geodesic pseudoconvex function. It can also be easily verified that $f(\bar{x}) \not< \mathscr{L}(\bar{x}, \alpha, \lambda)$. Thus, the weak duality theorem (Theorem 4.11) is verified.

The following theorem establishes the strong duality relating primal problem (MSIP) and (MSID$_\text{W}$).

Theorem 4.13 (Strong duality)**.** *Let $\bar{x}$ be a weakly efficient solution of (MSIP) such that Abadie constraint qualification (ACQ) is satisfied at $\bar{x}$. Then, there exists $\bar{\alpha} \in \mathbb{R}_+^m \setminus \{0\}$ and $\bar{\lambda} \in \mathscr{A}(\bar{x})$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_W$ and*

$$f(\bar{x}) = \mathscr{L}(\bar{x}, \bar{\alpha}, \bar{\lambda}).$$

Moreover, the following statements are true.

(i) *If the assumptions of weak duality (Theorem 4.11 (i)) hold true, then* $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ *is a weakly efficient solution of (MSID*$_W$*).*
(ii) *If the assumptions of weak duality (Theorem 4.11 (ii)) hold true, then* $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ *is an efficient solution of (MSID*$_W$*).*

Proof. Since, $\bar{x}$ is a weakly efficient solution of (MSIP) and (ACQ) is satisfied at $\bar{x}$, we infer from Theorem 3.1, that there exists $\bar{\alpha} \in \mathbb{R}^m_+ \setminus \{0\}$ satisfying $\sum_{i\in I} \bar{\alpha}_i = 1$, and $\bar{\lambda} \in \mathscr{A}(\bar{x})$, such that

$$\sum_{i\in I} \bar{\alpha}_i \operatorname{grad} f_i(\bar{x}) + \sum_{j\in J} \bar{\lambda}_j \operatorname{grad} g_j(\bar{x}) = 0.$$

Since $\bar{\lambda} \in \mathscr{A}(\bar{x})$, we have

$$\bar{\lambda}_j g_j(\bar{x}) = 0 \quad \forall j \in J,$$

and hence

$$\sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}) = 0.$$

Thus, we have

$$f(\bar{x}) = f(\bar{x}) + \sum_{j\in J} \lambda_j g_j(\bar{x}) = \mathscr{L}(\bar{x}, \bar{\alpha}, \bar{\lambda}).$$

That is, $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_{MW}$ and $f(\bar{x}) = \mathscr{L}(\bar{x}, \bar{\alpha}, \bar{\lambda})$.
(i) From weak duality theorem (Theorem 4.11 (i)), it follows that for any $(u, \alpha, \lambda) \in F_W$, we have

$$\mathscr{L}(u, \alpha, \lambda) \not< \mathscr{L}(\bar{x}, \bar{\alpha}, \bar{\lambda}).$$

This proves that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is a weakly efficient solution of (MSID$_\text{W}$).
(ii) From weak duality theorem (Theorem 4.11 (ii)), it follows that for any $(u, \alpha, \lambda) \in F_W$, we have

$$\mathscr{L}(u, \alpha, \lambda) \not\leq \mathscr{L}(\bar{x}, \bar{\alpha}, \bar{\lambda}).$$

This proves that $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an efficient solution of (MSID$_\text{W}$). □

The following theorem establishes the strict converse duality relating (MSIP) and (MSID$_\text{W}$).

Theorem 4.14 (Strict converse duality)**.** *Let* x^* *be a weakly efficient solution of (MSIP) such that Abadie constraint qualification (ACQ) is satisfied at* x^**. Let* $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ *be a weakly efficient solution of (MSID*$_W$*). If* $h := \sum_{i\in I} \alpha_i f_i + \sum_{j\in J} \lambda_j g_j$ *is geodesic strictly pseudoconvex at* $\bar{x}$*, then* $x^* = \bar{x}$*.*

Proof. If possible, let us assume that $x^* \neq \bar{x}$. Since x^* is a weakly efficient solution of (MSIP) and (ACQ) is satisfied at x^*, we can infer from Theorem 4.13 that there exist $\alpha^* \in \mathbb{R}^m_+ \setminus \{0\}$ and $\lambda^* \in \mathscr{A}(x^*)$ such that $(x^*, \alpha^*, \lambda^*) \in F_W$ and

$$f(x^*) = \mathscr{L}(x^*, \alpha^*, \lambda^*).$$

Further, it also follows from Theorem 4.13 that if h is a geodesic strictly pseudoconvex function, then $(x^*, \alpha^*, \lambda^*)$ is an efficient solution of $(\mathrm{MSID_W})$. Since $x^* \in F$ and $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_W$, then from Theorem 4.11(ii), it follows that

$$\mathscr{L}(\bar{x}, \bar{\alpha}, \bar{\lambda}) \prec f(x^*) = \mathscr{L}(x^*, \alpha^*, \lambda^*),$$

which is a contradiction. This completes the proof. □

We illustrate the results on Wolfe duality by the following example.

Example 4.15. Let us consider the problem $(\mathrm{P_2})$ on the Hadamard manifold $\mathscr{H}$ as considered in Example 4.6. Then the Wolfe dual problem related to (P_2) may be formulated as

$$\begin{aligned} (\mathrm{P_W}) \quad & \text{Maximize } \mathscr{L}(u, \lambda) = 2\sqrt{u_1} + \sqrt{u_2} + \sum_{j \in J} \lambda_j g_j(u) e, \\ & \text{subject to } \operatorname{grad} f(u) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(u) = (0,0), \end{aligned}$$

where $u \in \mathscr{H}$, $\lambda \in \mathbb{R}_+^{|J|}$, and $e = (1,1)$.
The feasible set of $(\mathrm{P_W})$ is given by

$$\begin{aligned} F_{\mathrm{W}} = \{ & (u, \lambda) \in \mathscr{H} \times \mathbb{R}_+^{|J|}, \\ & \operatorname{grad} f(u) + \sum_{j \in J} \lambda_j \operatorname{grad} g_j(u) = (0,0) \}. \end{aligned}$$

Let us consider the feasible point $\bar{x} = (e, e) \in F_{\mathrm{W}}$. Then, from Example 4.6, we observe that Abadie constraint qualification (ACQ) holds at $\bar{x}$ and $\bar{x}$ is an efficient solution of $(\mathrm{P_2})$. Thus, we see that all the assumptions for strong duality of Wolfe dual problem (Theorem 4.9) are satisfied.
Let $\bar{\lambda} : J \to \mathbb{R}$ be defined in the following manner

$$\bar{\lambda}_j = \begin{cases} 3\sqrt{e}, & \text{if } j = \frac{1}{3}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, it follows that

$$\bar{\lambda}_j g_j(\bar{x}) = 0, \quad \forall j \in J.$$

This implies there exist $\bar{\lambda} \in \mathscr{A}(\bar{x})$, such that

$$\begin{aligned} \operatorname{grad} f(\bar{x}) + \sum_{j \in J} \bar{\lambda}_j \operatorname{grad} g_j(\bar{x}) &= \left(e\sqrt{e}, \frac{e}{2}\sqrt{e} \right) + 3\sqrt{e} \left(-\frac{1 - \frac{1}{3}}{2} e, -\frac{\frac{1}{3}}{2} e \right) \\ &= \left(e\sqrt{e}, \frac{e}{2}\sqrt{e} \right) + 3\sqrt{e} \left(-\frac{1}{3} e, -\frac{1}{6} e \right) \\ &= (0,0). \end{aligned}$$

This shows that $(\bar{x}, \bar{\lambda}) \in F_W$.

$$f(\bar{x}) = f(\bar{x}) + \sum_{j\in J} \bar{\lambda}_j g_j(\bar{x}) = \mathscr{L}(\bar{x}, \bar{\lambda}).$$

Now, we observe that

$$f(x) = 2\sqrt{x_1} + \sqrt{x_2} = \frac{4\sqrt{x_1} + 2\sqrt{x_2}}{2} = \frac{f_1(x)}{2},$$
$$g_j(x) = \frac{1}{2} - \frac{1-j}{2}\ln x_1 - \frac{j}{2}\ln x_2 = \frac{1-(i-j)\ln x_1 - j\ln x_2}{2} = \frac{g'_j(x)}{2},$$

where,

$$f_1(x) = 4\sqrt{x_1} + 2\sqrt{x_2},$$
$$g'_j(x) = 1 - (i-j)\ln x_1 - j\ln x_2.$$

Let us consider that

$$\mathscr{L}(x, \lambda) = f(x) + \sum_{j\in J} \lambda_j g_j(x) = \frac{f_1(x) + \sum\limits_{j\in J} \lambda_j g'_j(x)}{2} = \frac{h(x)}{2},$$

where,

$$h(x) = \Big(f_1 + \sum_{j\in J} \lambda_j g'_j\Big)(x).$$

Then, we have the following

$$\nabla h(x) = \begin{pmatrix} \frac{2}{\sqrt{x_1}} - \sum\limits_{j\in J} \frac{1-j}{x_1} \\ \frac{1}{\sqrt{x_2}} - \sum\limits_{j\in J} \frac{j}{x_2} \end{pmatrix}, \quad \nabla^2 h(x) = \begin{pmatrix} -\frac{1}{x_1\sqrt{x_1}} + \sum\limits_{j\in J} \frac{1-j}{x_1^2} & 0 \\ 0 & -\frac{1}{2x_2\sqrt{x_2}} + \sum\limits_{j\in J} \frac{j}{x_2^2} \end{pmatrix}.$$

From the second-order covariant derivative, it follows that

$$\begin{aligned} H^g h(x) &= \nabla^2 h(x) - \nabla h(x)\Gamma \\ &= \begin{pmatrix} -\frac{1}{x_1\sqrt{x_1}} + \sum\limits_{j\in J} \frac{1-j}{x_1^2} & 0 \\ 0 & -\frac{1}{2x_2\sqrt{x_2}} + \sum\limits_{j\in J} \frac{j}{x_2^2} \end{pmatrix} \\ &\quad - \left[\left(\frac{2}{\sqrt{x_1}} - \sum_{j\in J} \frac{1-j}{x_1}\right)\begin{pmatrix} -\frac{1}{x_1} & 0 \\ 0 & 0 \end{pmatrix} + \left(\frac{1}{\sqrt{x_2}} - \sum_{j\in J} \frac{j}{x_2}\right)\begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{x_2} \end{pmatrix}\right] \\ &= \begin{pmatrix} \frac{1}{x_1\sqrt{x_1}} & 0 \\ 0 & \frac{1}{2x_2\sqrt{x_2}} \end{pmatrix}. \end{aligned}$$

Since $x_1, x_2 > 0$, hence, the (hyperbolic) Hessian, or the second-order covariant derivative of $h(x)$ is positive semidefinite. Thus, $h(x)$ is geodesic convex function.

Since, $\mathscr{L}(x, \lambda)$ is the ratio of a geodesic convex function $(h(x))$ and a positive affine function, this implies that $\left(f + \sum_{j \in J} \lambda_j g_j\right)(x)$ is a geodesic pseudoconvex function (see Theorem 2.8). Thus, we see that all the assumptions in Theorem 4.11 are satisfied. It can be verified that $(\bar{x}, \bar{\lambda})$ is a weakly efficient solution of the Wolfe dual problem $(\mathrm{P_W})$.

Remark 4.16. In view of Definition 10.1 in Udrişte [59] and Definition 13.2.1 in Rapcsák [46], every geodesic convex function is geodesic pseudoconvex and geodesic quasiconvex. Thus, the results presented in this paper generalize the corresponding results of optimality and duality from Tung and Tam [56].

5. Conclusion

In this paper, we have considered a class of multiobjective semi-infinite programming problems on Hadamard manifold (MSIP) and established the Karush-Kuhn-Tucker type sufficient optimality criteria for (MSIP) under generalized geodesic convexity assumptions. The sufficient optimality condition derived in this paper extend the sufficient optimality result derived by Tung and Tam [56] from geodesic convexity assumptions to geodesic pseudoconvexity and geodesic quasiconvexity assumptions. Moreover, related to (MSIP), we have formulated the Mond-Weir type dual problem $(\mathrm{MSIP_{MW}})$ and Wolfe type dual problem $(\mathrm{MSIP_{MW}})$ and derived the weak, strong and strict converse duality theorems. The weak and strong duality results derived in this paper extend the corresponding results of Tung and Tam [56] from geodesic convexity assumptions to geodesic pseudoconvexity and geodesic quasiconvexity assumptions. In particular, the results of the paper generalize some other well known results in $\mathbb{R}^n$, see for instance, [3, 34, 35, 36]. Several non-trivial examples have been given to illustrate the significance of these results. Our work in this paper leaves various avenues for future research. For example, it would be interesting to extend the results in this paper for non-smooth multiobjective semi-infinite problems on Hadamard manifolds. Further, we intend to investigate multiobjective semi-infinite problems on Hadamard manifolds with uncertain data in objective functions.

References

[1] P.A. Absil, C.G. Baker and K.A. Gallivan, Trust-region methods on Riemannian manifolds. *Found. Comput. Math.* **7** (2007) 303-330.

[2] P.A. Absil, R. Mahony and R. Sepulchre, *Optimization Algorithms on Matrix Manifolds.* Princeton University Press, Princeton, NJ (2009).

[3] K.J. Arrow and A.C. Enthoven, Quasi-concave programming. *Econometrica: Journal Of The Econometric Society.* **29** (1961) 779-880.

[4] A. Barani and S. Hosseini, Characterization of solution sets of convex optimization problems in Riemannian manifolds. *Arch. Math. (Basel).* **114** (2020) 215-225.

[5] A. Barani, On pseudoconvex functions in Riemanian manifolds. *J. Finsler Geom. Appl.* **2**, (2021) 14-22.
[6] A. Barani, Convexity of the solution set of a pseudoconvex inequality in Riemannian manifolds. *Numer. Funct. Anal. Optim.* **39** (2018) 588-599.
[7] G.C. Bento and J.G. Melo, Subgradient method for convex feasibility on Riemannian manifolds. *J. Optim. Theory Appl.* **152** (2012) 773-785.
[8] R. Bergmann and R. Herzog, Intrinsic formulation of KKT conditions and constraint qualifications on smooth manifolds. *SIAM J. Optim.* **29** (2019) 2423-2444.
[9] J. Borwein and A.S. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples.* Springer Science & Business Media, NY (2010).
[10] N. Boumal, B. Mishra, P.A. Absil and R. Sepulchre. Manopt, a matlab toolbox for optimization on manifolds. *J. Mach. Learn. Res.* **15** (2014) 1455-1459.
[11] A. Charnes, W. Cooper and K. Kortanek, Duality, Haar programs, and finite sequence spaces. *Proc. Natl. Acad. Sci. USA.* **48** (1962) 783.
[12] A. Charnes, W. Cooper and K. Kortanek, Duality in semi-infinite programs and some works of Haar and Carathéodory. *Manag. Sci.* **9** (1963) 209-228.
[13] A. Charnes, W. Cooper and K. Kortanek, On the theory of semi-infinite programming and a generalization of the Kuhn-Tucker saddle point theorem for arbitrary convex functions. *Naval Res. Logist. Quart.* **16** (1969) 41-52.
[14] S.L. Chen, Existence results for vector variational inequality problems on Hadamard manifolds. *Optim. Lett.* **14** (2020) 2395-2411.
[15] S.L. Chen, The KKT optimality conditions for optimization problem with interval-valued objective function on Hadamard manifolds. *Optimization.* (2020).
[16] T.D. Chuong and D.S. Kim, Nonsmooth semi-infinite multiobjective optimization problems. *J. Optim. Theory Appl.* **160** (2014) 748-762.
[17] J.X. Da Cruz Neto, O.P. Ferreira, L.R. Lucambio Perez and S.Z. Németh, Convex-and monotone-transformable mathematical programming problems and a proximal-like point method. *J. Global Optim.* **35**(1), (2006) 53-69.
[18] M.P. Do Carmo, *Riemannian Geometry.* Springer (1992).
[19] I. Ekeland, On the variational principle. *J. Math. Anal. Appl.* **47** (1974) 324-353.
[20] X. Gao, Necessary optimality and duality for multiobjective semi-infinite programming. *J. Theor. Appl. Inf. Technol.* **46** (2012) 347-354.
[21] X. Gao, Optimality and duality for non-smooth multiple objective semi-infinite programming. *J. Netw.* **8** (2013).
[22] M.A. Goberna and M.A. Lopez, Linear semi-infinite programming theory: An updated survey. *European J. Oper. Res.* **143** (2002) 390-405.
[23] M.A. Goberna and M.A. López, Recent contributions to linear semi-infinite optimization: an update. *Ann. Oper. Res.* **271** (2018) 237-278.
[24] A. Haar, Über lineare ungleichungen. *Acta Sci. Math. (Szeged).* **2** (1924) 1-14.
[25] J. Jost, *Riemannian Geometry and Geometric Analysis.* Springer (2008).

[26] N. Kanzi and S. Nobakhtian, Optimality conditions for non-smooth semi-infinite programming. *Optimization.* **59** (2010) 717-727.

[27] N. Kanzi and S. Nobakhtian, Optimality conditions for nonsmooth semi-infinite multiobjective programming. *Optim. Lett.* **8** (2014) 1517-1528.

[28] M.M. Karkhaneei and N. Mahdavi-Amiri, Nonconvex weak sharp minima on Riemannian manifolds. *J. Optim. Theory Appl.* **183** (2019) 85-104.

[29] O. Kostyukova and T. Tchemisova, Optimality conditions for convex semi-infinite programming problems with finitely representable compact index sets. *J. Optim. Theory Appl.* **175** (2017) 76-103.

[30] J.M. Lee, *Introduction to Riemannian Manifolds.* Springer (2018).

[31] C. Li, B.S. Mordukhovich, J. Wang and J.C. Yao, Weak sharp minima on Riemannian manifolds. *SIAM J. Optim.* **21** (2011) 1523-1560.

[32] M.A. López and E. Vercher, Optimality conditions for nondifferentiable convex semi-infinite programming. *Math. Program.* **27** (1983) 307-319.

[33] D. T. Luc, *Theory of Vector Optimization.* Springer (1989).

[34] O.L. Mangasarian, Nonlinear Programming. SIAM, (1994).

[35] O.L. Mangasarian, Pseudo-convex functions. *J. SIAM Control Ser. A.* **3** (1965) 281-290.

[36] O.L. Mangasarian, Duality in nonlinear programming. *Quart. Appl. Math.* **20** (1962) 300-302.

[37] S.K. Mishra and B.B. Upadhyay, *Pseudolinear Functions and Optimization.* Chapman and Hall/CRC (2019).

[38] S.K. Mishra, M. Jaiswal and L.T.H. An, Duality for nonsmooth semi-infinite programming problems. *Optim. Lett.* **6** (2012) 261-271.

[39] S. Németh, Five kinds of monotone vector fields. *Pure Appl. Math.* **9** (1999) 417–428.

[40] T.H. Pham, Optimality conditions and duality for multiobjective semi-infinite programming with data uncertainty via Mordukhovich subdifferential. *Yugosl. J. Oper. Res.* **31** (2021) 495-514.

[41] E.A. Papa Quiroz, N. Baygorrea Cusihuallpa and N. Maculan, Inexact Proximal Point Methods for Multiobjective Quasiconvex Minimization on Hadamard Manifolds. *J. Optim. Theory Appl.* **186** (2020) 879-898.

[42] E.A. Papa Quiroz, E.M. Quispe and P.R. Oliveira, Steepest descent method with a generalized Armijo search for quasiconvex functions on Riemannian manifolds. *J. Math. Anal. Appl.* **341** (2008) 467–477.

[43] E.A. Papa Quiroz and P.R. Oliveira, Proximal point methods for quasiconvex and convex functions with Bregman distances on Hadamard manifolds. *J. Convex Anal.* **16**(1), (2009) 49-69.

[44] E.A. Papa Quiroz and P.R. Oliveira, Full convergence of the proximal point method for quasiconvex functions on Hadamard manifolds. *ESAIM: Control. Optim. Cal. Var.* **18**, (2012) 483-500.

[45] M. Rahimi and M. Soleimani-Damaneh, Isolated efficiency in nonsmooth semi-infinite multi-objective programming. *Optimization.* **67** (2018) 1923-1947.

[46] T. Rapcsák, *Smooth Nonlinear Optimization in* $\mathbb{R}^n$. Springer Science & Business Media (2013).

[47] G. Ruiz-Garzón, R. Osuna-Gómez and J. Ruiz-Zapatero, Necessary and sufficient optimality conditions for vector equilibrium problems on Hadamard manifolds. *Symmetry.* **11** (2019) 1037.

[48] T. Sakai, Riemannian Geometry. American Mathematical Society, (1996).

[49] A. Shahi and S.K. Mishra, On geodesic convex and generalized geodesic convex functions over Riemannian manifolds. *AIP Conf. Proc.* (2018) 030006, doi: https://doi.org/10.1063/1.5042176.

[50] O. Stein and G. Still, Solving semi-infinite optimization problems with interior point techniques. *SIAM J. Control Optim.* **42** (2003) 769-788.

[51] G.J. Tang and N.J. Huang, Korpelevich's method for variational inequality problems on Hadamard manifolds. *J. Global Optim.* **54** (2012) 493-509.

[52] G.J. Tang, L.W. Zhou and N.J. Huang, The proximal point algorithm for pseudomonotone variational inequalities on Hadamard manifolds. *Optim. Lett.* **7** (2013) 779-790.

[53] S. Treanţă, P. Mishra and B.B. Upadhyay, Minty Variational Principle for Nonsmooth Interval-Valued Vector Optimization Problems on Hadamard Manifolds. *Mathematics.* **10**, (2022) 523.

[54] L.T. Tung, Karush-Kuhn-Tucker optimality conditions and duality for convex semi-infinite programming with multiple interval-valued objective functions. *J. Appl. Math. Comput.* **62** (2020) 67-91.

[55] L.T. Tung, Karush-Kuhn-Tucker optimality conditions and duality for multiobjective semi-infinite programming with vanishing constraints. *Ann. Oper. Res.* (2020) 1-28.

[56] L.T. Tung and D.H. Tam, Optimality conditions and duality for multiobjective semi-infinite programming on Hadamard manifolds. *Bull. Iranian Math. Soc.* (2021) 1-29.

[57] B.B. Upadhyay, S.K. Mishra and S.K. Porwal, Explicitly geodesic B-preinvex functions on Riemannian Manifolds. *Trans. Math. Program. Appl.* **2** (2015) 1-14.

[58] B.B. Upadhyay, I.M. Stancu Minasian, P. Mishra and R.N. Mohapatra, On Generalized Vector Variational Inequalities and Nonsmooth Vector Optimization Problems on Hadamard Manifolds involving Geodesic Approximate Convexity. *Adv. Nonlinear Var. Inequal.* **25** (2022) 1-25.

[59] C. Udrişte, *Convex Functions and Optimization Methods on Riemannian Manifolds.* Springer Science & Business Media (2013).

[60] T. Weir and B. Mond, Generalised convexity and duality in multiple objective programming. *Bull. Aust. Math. Soc.* **39** (1989) 287-299.

[61] P. Wolfe, A duality theorem for non-linear programming. *Quart. Appl. Math.* **19** (1961) 239-244.

[62] W.H. Yang, L.H. Zhang and R. Song, Optimality conditions for the nonlinear programming problems on Riemannian manifolds. *Pac. J. Optim.* **10** (2014) 415-434.

[63] Q. Zhang, Optimality conditions and duality for semi-infinite programming involving B-arcwise connected functions. *J. Global Optim.* **45** (2009) 615-629.