

A PATH-FOLLOWING INTERIOR-POINT ALGORITHM FOR MONOTONE LCP BASED ON A MODIFIED NEWTON SEARCH DIRECTION

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Abstract. In this paper, we propose a short-step feasible full-Newton step path-following interior-point algorithm (IPA) for monotone linear complementarity problems (LCPs). The proposed IPA uses the technique of algebraic equivalent transformation (AET) induced by an univariate function to transform the centering equations which defines the central-path. By applying Newton's method to the modified system of the central-path of LCP, a new Newton search direction is obtained. Under new appropriate defaults of the threshold τ which defines the size of the neighborhood of the central-path and of θ which determines the decrease in the barrier parameter, we prove that the IPA is well-defined and converges locally quadratically to a solution of the monotone LCPs. Moreover, we derive its iteration bound, namely, $\mathcal{O}(\sqrt{n} \log \frac{n}{\epsilon})$ which coincides with the best-known iteration bound for such algorithms. Finally, some numerical results are presented to show its efficiency.

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1. INTRODUCTION

After the seminal work of Karmarkar [25] for linear optimization (LO), interior-point methods revitalized as an active area of research in mathematical programming. Among them the class of path-following primal-dual IPAs deserved much more attention due to their polynomial complexity and their practical efficiency (see *e.g.*, [2, 3, 7, 22, 23, 30, 34]). Determining a search direction plays an important role in IPAs. In the last decade, several types of search directions have been proposed. Some of them are based on the strategy of so-called self-regular and kernel barrier functions (see *e.g.*, [4, 5, 8, 9, 33]). Meanwhile, others are based on the strategy of the AET technique applied to the centering equation of the system which characterizes the central-path. In 2003, Darvay [13], uses the AET technique based on the univariate function $\psi(t) = \sqrt{t}$ for LO. By means of this function, a new type of Newton direction is obtained and the best iteration bound for feasible short-step IPAs is derived. This method was extended successfully to convex quadratic optimization (CQO) and monotone LCP by Achache (see *e.g.*, [1–3]), semidefinite optimization (SDO) and second-order cone optimization (SCOP) by Wang and Bai [35, 36]. Besides, Kheirfam and Haghighi [28] investigated the AETs based on the function $\psi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ to solve $P_*(\kappa)$ -LCPs, where the best iteration bound is achieved for short-step methods. Subsequently, Haddou

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et al. [24] presented a generalized direction in interior-point methods for monotone LCP. Their approach is based on AET induced by the class of smooth concave univariate functions. By utilizing the AET technique based on the logarithmic function $\psi(t) = \log t$, Pan *et al.* [32] presented an infeasible IPA to solve LO. Furthermore, Darvay and Takács [15] developed a new IPA for LO based on a new modified search direction induced by an asymptotic barrier kernel function. The best polynomial complexity is provided. Recently, Darvay *et al.* [16], proposed an IPA for LO where their search direction is based on AET introduced by the new univariate function $\psi(t) = t - \sqrt{t}$. Later, Darvay *et al.* [17] generalized this algorithm for sufficient LCP. Also we mention that Fischer proposed a damped Newton-type IPA for monotone LCP [20]. Here he reformulated the LCP as an equivalent nonlinear system of equations based on the so-called NCP-functions. Under some conditions, the super-linear convergence of this algorithm is established.

In [29], Kheirfam and Nasrollahi utilized the AETs technique based on the power function $\psi_q(t) = t^{\frac{q}{2}}, q \geq 1$, to develop a full-Newton short-step IPA for LO. They offered a family of new search directions with respect to the parameter q . Further, under defaults $\tau = \frac{2}{(q-1)^2+2}$ and $\theta = \frac{1}{((2q-1)^3-5q)\sqrt{n}}$ with $q \geq 3$, they obtained the following iteration bound, namely, $\mathcal{O}\left(\sqrt{n} \log \frac{\mu^0 \left(n + \frac{4q(q-2)}{((q-1)^2+2)^2}\right)}{\epsilon}\right)$. Their study includes some earlier works. For example,

- for $q = 1$, then $\psi(t) = \sqrt{t}$ and which is introduced by Darvay in [14, 18] for LO and $P_*(\kappa)$ -LCP;
- for $q = 2$, then $\psi(t) = t$, this gives the classical Newton directions [34];
- for $q = 3$, then $\psi(t) = t^{\frac{3}{2}}$ which is introduced recently by Moussaoui and Achache in [31] for CQO;
- for $q = 4$, then $\psi(t) = t^2$ which is introduced by Kheirfam in [27] for LO.

For more details about the AETs technique we direct the reader to the papers (see *e.g.*, [14, 18, 26, 37]) and the references therein.

Based on the iteration bound obtained by [29], we notice that if q becomes very large then θ gets very small. Consequently, the rate $(1 - \theta)$ which determines the decrease in the barrier parameter converges to one. This leads to a slow convergence and even to a divergence of their algorithm. So it clear that taking a member of $\psi_q(t) = t^{\frac{q}{2}}$ with a large value of q leads to bad numerical results.

In this paper, in order to improve the numerical results of these algorithms, we reconsider the analysis of their IPAs designed for LO to monotone LCP where a non parametric univariate function, namely, $\psi_q(t) = t^{\frac{5}{2}}$ is suggested. Therefore, similar to LO, we use the AET technique introduced by this function to nonlinear equations of the system which defines the central-path of monotone LCPs, a modified nonlinear equations is obtained. The application of Newton method to the latter, a modified search direction is offered. The proposed IPA uses full-Newton steps for tracing approximately the central-path. Unlike LO case, the presence of non orthogonality of scaled directions in LCPs, a different analysis is stated. Further, under new appropriate choices of defaults τ and θ , we prove that the algorithm is well-defined and converges locally quadratically to a solution of monotone LCP (these results are based on the useful Lemma 4.4). Moreover, the currently best known iteration bound for the algorithm with short-update method, namely, $\mathcal{O}\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$ is obtained. This complexity is analogue to those achieved by many authors (see *e.g.* [1–3, 14, 18, 26, 27, 37]). Some numerical results are presented to evaluate our proposed algorithm. In addition, to accelerate the speed of convergence of our original algorithm, some relaxations are imported on the selection of the default θ . Finally, we compare the performances of our algorithm with a previously Fischer type IPAs on a set of monotone LCPs.

The outline of the paper is as follows. In Section 2, preliminaries notions and the problem description are presented. In Section 3, the modified search directions based on AETs for the centering equation is discussed. Moreover, the generic feasible short-update full-Newton step IPA for monotone LCPs is presented. In Section 4, the analysis of the algorithm is given. Further, its iteration bound is obtained. In Section 5, some numerical results are reported. Also for the performances of our algorithm, some modifications are suggested. In the last section we present a general conclusion of the work carried out, some remarks, as well as some perspectives and suggestions for future work.

NOTATIONS

The following notations are used throughout the paper. Given $x, y \in \mathbb{R}^n$, $x^T y = \sum_{i=1}^n x_i y_i$ is their usual scalar product whereas $xy = (x_i y_i)_{1 \leq i \leq n}$ denotes their Hadamard product and the same as for the vectors $x/y = (x_i/y_i)_{1 \leq i \leq n}$, where $y_i \neq 0, \forall i$, $\sqrt{x} = (\sqrt{x_i})_{1 \leq i \leq n}$ and $x^{-1} = (1/x_i)_{1 \leq i \leq n}$. The identity matrix and the vector of all ones are denoted by I and e , respectively. Moreover, $\text{diag}(x)$ denotes a diagonal matrix, which contains on his main diagonal the components of the vector x . For $v \in \mathbb{R}^n$, $v \geq 0$ means that $v_i \geq 0$ for all $i = 1, \dots, n$ and $\min v = \min_{1 \leq i \leq n} v_i$. Let $x \in \mathbb{R}^n$, $\|x\|$ and $\|x\|_\infty$ denote its Euclidean norm and its maximum norm, respectively. If $f(t)$ and $g(t)$ are two positive real valued functions, then $f(t) = \mathcal{O}(g(t))$ if there exists a positive constant c so that $f(t) \leq cg(t)$.

2. PRELIMINARIES AND THE PROBLEM STATEMENT

A monotone LCP consists in finding a couple of vectors $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$y - Mx = q, \quad xy = 0, \quad x \geq 0, \quad y \geq 0, \quad (1)$$

where $M \in \mathbb{R}^{n \times n}$ is positive semi-definite (PSD) and $q \in \mathbb{R}^n$. Throughout the paper, we assume that the interior-point condition (IPC) holds for monotone LCP (1), *i.e.*, there exists a pair of vectors (x^0, y^0) such that

$$y^0 - Mx^0 = q, \quad x^0 > 0, \quad y^0 > 0.$$

In this case the monotone LCP (1) has a solution. For more comprehension of LCP, we recommended the monograph of Cottle *et al.* [11].

The main idea of path-following IPAs is to replace the equation $xy = 0$ in (1) by the parameterized equation $xy = \mu e$ where $\mu > 0$. Hence, we obtain the system of equations:

$$y - Mx = q, \quad xy = \mu e, \quad x, y \geq 0. \quad (2)$$

Under the IPC condition, Kojima *et al.* [30] shows that system (2) has a unique solution denoted by $(x(\mu), y(\mu))$ for each $\mu > 0$, which is called the μ -center of monotone LCP. The set of μ -centers is called the central-path of monotone LCP. If μ goes to zero, then the limit of central-path exists and since the limit point satisfies the complementarity condition $xy = 0$, the limit yields a solution of LCP. Applying Newton's method to system (2) for a given strictly feasible point (x, y) , *i.e.*, the IPC holds, then the Newton direction $(\Delta x, \Delta y)$ at this point is the unique solution of the linear system of equations:

$$\begin{cases} \Delta y - M\Delta x = 0, \\ x\Delta y + y\Delta x = \mu e - xy. \end{cases} \quad (3)$$

The system (3) gives the classical Newton search direction for LCP [30, 34, 41].

3. THE MODIFIED SEARCH DIRECTIONS FOR LCP

Following [2, 13], the AET technique for computing a new search direction for IPAs is based on the transformation of the centrality equation $xy = \mu e$ in (2) to the new equation $\psi\left(\frac{xy}{\mu}\right) = \psi(e)$ where $\psi : (0, +\infty) \rightarrow \mathbb{R}$ is a continuously differentiable function and invertible, *i.e.*, ψ^{-1} exists. Then, (2) is transformed to the following system:

$$y - Mx = q, \quad \psi\left(\frac{xy}{\mu}\right) = \psi(e), \quad x, y \geq 0, \quad (4)$$

where ψ is applied coordinate-wisely. Applying Newton's method to system (4) for a given strictly feasible point (x, y) yields the new Newton system:

$$\begin{cases} \Delta y - M\Delta x = 0, \\ \frac{1}{\mu}y\psi'\left(\frac{xy}{\mu}\right)\Delta x + \frac{1}{\mu}x\psi'\left(\frac{xy}{\mu}\right)\Delta y = \psi(e) - \psi\left(\frac{xy}{\mu}\right), \end{cases} \quad (5)$$

where ψ' denotes the derivative of ψ .

Next, to facilitate the analysis of the algorithm, we introduce the following notations:

$$v := \sqrt{\frac{xy}{\mu}}, \quad d_x := \frac{v\Delta x}{x}, \quad d_y := \frac{v\Delta y}{y}, \quad d := \sqrt{\frac{x}{y}}. \quad (6)$$

From (3) and (6), system (5) can be written as follows:

$$\begin{cases} d_y - \bar{M}d_x = 0, \\ d_x + d_y = p_v, \end{cases} \quad (7)$$

where

$$p_v := \frac{\psi(e) - \psi(v^2)}{v\psi'(v^2)}$$

and $\bar{M} := DMD$, $D := \text{diag}(d)$. The system (7) determines a family of new scaled Newton search directions related to the function ψ .

Now, we consider the AET introduced by the function $\psi(t) = t^{\frac{q}{2}}$ with $q = 5$. This yields

$$p_v = \frac{2}{5}(v^{-4} - v). \quad (8)$$

Moreover, system (5) becomes

$$\begin{cases} \Delta y - M\Delta x = 0, \\ y\Delta x + x\Delta y = \frac{2\mu}{5} \left(\left(\frac{xy}{\mu} \right)^{\frac{-3}{2}} - \left(\frac{xy}{\mu} \right) \right). \end{cases} \quad (9)$$

Next, according to (8), we define a norm-based proximity measure as follows:

$$\delta(v) := \delta(xy; \mu) := \frac{5}{2} \|p_v\| = \|v^{-4} - v\|. \quad (10)$$

Clearly, $\delta(v) = 0 \Leftrightarrow v = e \Leftrightarrow xy = \mu e$. Therefore, the value of $\delta(v)$ can be considered as a measure of the distance between the given pair (x, y) and the central-path. Furthermore, we define the τ -neighborhood of the central-path as follows:

$$\mathcal{N}(\tau, \mu) = \{(x, y) : x > 0, y = Mx + q > 0 : \delta(xy; \mu) \leq \tau\},$$

where τ is a threshold (default) and $\mu > 0$, is fixed.

3.1. The Algorithm

Now we are ready to describe the generic full-Newton step IPA for monotone LCP as follows. First, we use a suitable threshold value, with $0 < \tau < 5$ and we suppose that an initial point $(x^0, y^0) \in \mathcal{N}(\tau, \mu_0)$ exists for certain $\mu_0 > 0$ is known. The full-Newton step between successive iterates is defined as $(x_+, y_+) = (x + \Delta x, y + \Delta y)$ where the Newton directions Δx and Δy are solutions for linear system (9). Then it updates the parameter μ by the factor $(1 - \theta)$ with $0 < \theta < 1$, and target a new μ -center and so on. This procedure is repeated until the stopping criterion $x^T y \leq \epsilon$ is satisfied for a given accuracy parameter ϵ .

Therefore, the generic algorithm the full-Newton step IPA for monotone LCP is stated in Figure 1 as follows.

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Input
  An accuracy parameter  $\epsilon > 0$ ;
  a barrier update parameter  $\theta$ ,  $0 < \theta < 1$ , (default  $\theta = \frac{1}{35\sqrt{2n}}$ );
  a threshold parameter  $0 < \tau < 1$ , (default  $\tau = \frac{1}{4}$ );
  an initial point  $(x^0, y^0)$  and  $\mu > 0$ , s.t.  $(x^0, y^0) \in \mathcal{N}(\tau, \mu)$ ;
begin
   $x := x^0$ ;  $y := y^0$ ;
  while  $n\mu > \epsilon$  do
     $\mu := (1 - \theta)\mu$ ;
    compute  $(\Delta x, \Delta y)$  from (9);
    set  $x := x + \Delta x$ ;  $y := y + \Delta y$ ;
  end while
end.

```

FIGURE 1. Generic full-Newton step IPA for monotone LCP.

4. ANALYSIS OF THE ALGORITHM

In this section, we are going to show across our new selecting defaults of θ and τ , described in Figure 1, that Algorithm 3.1 is well-defined and converges locally quadratically to a solution of monotone LCP. Moreover, we prove that our algorithm solves the monotone LCP in polynomial time. We mention that due to the non orthogonality of the scaled direction, our analysis is quite different from those used in LO case.

Next, the following technical results are fundamental tools in our analysis.

Lemma 4.1. *Let $\mu > 0$ and (d_x, d_y) be a solution of system (7) with $\delta := \delta(xy; \mu)$. Then one has*

$$0 \leq d_x^T d_y \leq \frac{2}{25} \delta^2 \quad (11)$$

and

$$\|d_x d_y\|_\infty \leq \frac{\delta^2}{25}, \quad \|d_x d_y\| \leq \frac{\sqrt{2}}{25} \delta^2. \quad (12)$$

Proof. For the first part of (11) we have, $d_x^T d_y = \frac{1}{\mu} (\Delta x)^T \Delta y = \frac{1}{\mu} (\Delta x)^T M \Delta x \geq 0$, since M is PSD. The second part of it, follows trivially from the following equality

$$\frac{4}{25} \delta^2 = \|p_v\|^2 = \|d_x + d_y\|^2 = \|d_x\|^2 + \|d_y\|^2 + 2d_x^T d_y \geq 2d_x^T d_y.$$

For the first claim in (12), since

$$d_x d_y = \frac{1}{4} ((d_x + d_y)^2 - (d_x - d_y)^2),$$

and

$$\|d_x + d_y\|^2 = \|d_x - d_y\|^2 + 4d_x^T d_y,$$

but since $d_x^T d_y \geq 0$, it follows that $\|d_x - d_y\| \leq \|d_x + d_y\|$. On the other hand,

$$\begin{aligned} \|d_x d_y\|_\infty &= \frac{1}{4} \left\| (d_x + d_y)^2 - (d_x - d_y)^2 \right\|_\infty \\ &\leq \frac{1}{4} \max(\|d_x + d_y\|_\infty^2, \|d_x - d_y\|_\infty^2) \\ &\leq \frac{1}{4} \max(\|d_x + d_y\|^2, \|d_x - d_y\|^2) \leq \frac{1}{4} \|d_x + d_y\|^2 = \frac{1}{4} \|p_v\|^2 = \frac{\delta^2}{25}. \end{aligned}$$

Therefore, $\|d_x d_y\|_\infty \leq \frac{\delta^2}{25}$. Next, to prove the last claim of (12), we have

$$\begin{aligned}\|d_x d_y\|^2 &= e^T (d_x d_y)^2 = \frac{1}{16} e^T \left((d_x + d_y)^2 - (d_x - d_y)^2 \right)^2 \\ &= \frac{1}{16} \left\| (d_x + d_y)^2 - (d_x - d_y)^2 \right\|^2 \\ &\leq \frac{1}{16} \left(\|(d_x + d_y)^2\|^2 + \|(d_x - d_y)^2\|^2 \right) \leq \frac{1}{16} (\|d_x + d_y\|^4 + \|d_x - d_y\|^4) \\ &\leq \frac{1}{8} \|d_x + d_y\|^4 = \frac{1}{8} \|p_v\|^4 = \frac{2}{5^4} \delta^4.\end{aligned}$$

Hence, $\|d_x d_y\| \leq \frac{\sqrt{2}}{25} \delta^2$. This completes the proof. $\square$

In the next lemma, we show the feasibility of a full-Newton step under the condition $\delta < 5$ throughout the algorithm.

Lemma 4.2. *Let $\delta < 5$, then $x_+ = x + \Delta x > 0$ and $y_+ = y + \Delta y > 0$, i.e., the full-Newton step is strictly feasible.*

Proof. For any $\alpha \in [0, 1]$ and (x, y) a strictly feasible point for LCP (1), we have

$$x(\alpha)y(\alpha) = (x + \alpha\Delta x)(y + \alpha\Delta y) = xy + \alpha(x\Delta y + y\Delta x) + \alpha^2\Delta x\Delta y.$$

Using (6), we get,

$$x(\alpha)y(\alpha) = \mu \left((1 - \alpha)v^2 + \alpha(v^2 + vp_v + \alpha d_x d_y) \right). \quad (13)$$

So $x(\alpha)y(\alpha) > 0$ if $v^2 + vp_v + \alpha d_x d_y > 0$. Due to (12) in Lemma 4.1 and (8) with $\delta < 5$, it follows that

$$\begin{aligned}v^2 + vp_v + \alpha d_x d_y &\geq v^2 + vp_v - \alpha \|d_x d_y\|_\infty e \\ &\geq v^2 + vp_v - \alpha \frac{\delta^2}{25} e \\ &= \frac{3}{5}v^2 + \frac{2}{5}v^{-3} - \alpha \frac{\delta^2}{25} e \\ &> \frac{3}{5}v^2 + \frac{2}{5}v^{-3} - e.\end{aligned}$$

Hence it is clear that $x(\alpha)y(\alpha) > 0$ if $\frac{3}{5}v^2 + \frac{2}{5}v^{-3} - e \geq 0$. Let us consider the function $g(t) = \frac{3}{5}t^2 + \frac{2}{5}t^{-3} - 1$ for $t > 0$. From $g''(t) > 0$, it follows that g is strictly convex and therefore, it has a minimum at $t = 1$, since $g'(1) = 0$, so $g(t) \geq g(1) = 0$, $\forall t > 0$. Hence,

$$\frac{3}{5}v^2 + \frac{2}{5}v^{-3} - e \geq 0. \quad (14)$$

Thus $x(\alpha)y(\alpha) > 0$ for $\alpha \in [0, 1]$. Since x and y are positive we obtain that $x(\alpha) > 0$ and $y(\alpha) > 0$ for all $\alpha \in [0, 1]$. Then, by continuity the vectors $x(1) = x_+$ and $y(1) = y_+$ are positive, i.e., $x_+ > 0$ and $y_+ > 0$ are strictly feasible. This completes the proof. $\square$

For convenience, we may write

$$v_+ = \sqrt{\frac{x_+ y_+}{\mu}}.$$

Lemma 4.3. *If $\delta < 5$, then $\min v_+ \geq \frac{1}{5}\sqrt{25 - \delta^2}$.*

Proof. From (13) in the proof of Lemma 4.2, setting $\alpha = 1$ and as $x(1) = x_+$, $y(1) = y_+$ and $p_v = \frac{2}{5}(v^{-4} - v)$, we get

$$v_+^2 = v^2 + vp_v + d_x d_y = \frac{3}{5}v^2 + \frac{2}{5}v^{-3} + d_x d_y.$$

By the inequality 14, we have seen that $\frac{3}{5}v^2 + \frac{2}{5}v^{-3} - e \geq 0$ if $\delta < 5$, hence $\frac{3}{5}v^2 + \frac{2}{5}v^{-3} \geq e$. Consequently, $v_+^2 \geq e + d_x d_y$. Next, using (12), we deduce that

$$v_+^2 \geq e + d_x d_y \geq (e - \|d_x d_y\|_\infty e) \geq \left(1 - \frac{\delta^2}{25}\right) e = \frac{1}{25}(25 - \delta^2)e,$$

hence $\min v_+ \geq \frac{1}{5}\sqrt{25 - \delta^2}$. This completes the proof. $\square$

Next, we prove that the iterate across the proximity measure is locally quadratically convergent during the algorithm.

Lemma 4.4. *Let (x, y) a strictly feasible point for LCP (1) and $\delta < 5$. Then*

$$\delta^+ := \delta(v_+) := \delta(x_+ y_+; \mu) \leq \left(\frac{5^2}{25 - \delta^2} + \frac{5^4}{(25 - \delta^2)^2} + \frac{5}{5 + \sqrt{25 - \delta^2}} \right) \left(\frac{3}{5} + \frac{\sqrt{2}}{25} \right) \delta^2.$$

In addition, assume $\delta \leq \frac{1}{4}$, then $\delta^+ \leq \delta^2$, which means the locally quadratically convergence of the proximity measure.

Proof. We have

$$\delta^+ = \|v_+^{-4} - v_+\| = \left\| \frac{e - v_+^5}{v_+^4} \right\| = \left\| (e - v_+^2) \left(\frac{e + v_+ + v_+^2 + v_+^3 + v_+^4}{(e + v_+)v_+^4} \right) \right\|.$$

Consider the function

$$f(t) = \frac{1 + t + t^2 + t^3 + t^4}{(1 + t)t^4} = \frac{1}{t^2} + \frac{1}{t^4} + \frac{1}{1 + t}.$$

Using f we can write

$$\delta^+ = \|f(v_+)(e - v_+^2)\| \leq \|f(v_+)\|_\infty \|e - v_+^2\|.$$

The function f is continuous and positive on $(0, +\infty)$. Moreover, from $f'(t) < 0$ for $t > 0$ it follows that f is decreasing, therefore,

$$0 < |f((v_+)_i)| = f((v_+)_i) \leq f(\min v_+) \leq f\left(\frac{1}{5}\sqrt{25 - \delta^2}\right).$$

Consequently

$$\|f(v_+)\|_\infty = \frac{5^2}{25 - \delta^2} + \frac{5^4}{(25 - \delta^2)^2} + \frac{5}{5 + \sqrt{25 - \delta^2}}.$$

This implies that

$$\delta^+ \leq \left(\frac{5^2}{25 - \delta^2} + \frac{5^4}{(25 - \delta^2)^2} + \frac{5}{5 + \sqrt{25 - \delta^2}} \right) \|e - v_+^2\|.$$

Next, setting $\alpha = 1$ in (13), and due to (8), we have,

$$\|e - v_+^2\| = \|e - (v^2 + vp_v + d_x d_y)\| = \left\| e - \frac{3}{5}v^2 - \frac{2}{5}v^{-3} - d_x d_y \right\|.$$

Hence

$$\|e - v_+^2\| \leq \left\| e - \frac{3}{5}v^2 - \frac{2}{5}v^{-3} \right\| + \|d_x d_y\|.$$

Next, we may write

$$\left\| e - \frac{3}{5}v^2 - \frac{2}{5}v^{-3} \right\| = \left\| \varphi(v) \cdot \frac{25}{4}p_v^2 \right\|,$$

where

$$\varphi(v) = \frac{e - \frac{3}{5}v^2 - \frac{2}{5}v^{-3}}{(v^{-4} - v)^2} = -\frac{v^5(3v^3 + 6v^2 + 4v + 2e)}{5(v^4 + v^3 + v^2 + v + e)^2}.$$

Let us consider the function

$$\varphi(t) = -\frac{t^5(3t^3 + 6t^2 + 4t + 2)}{5(t^4 + t^3 + t^2 + t + 1)^2}. \quad (15)$$

This function is continuous and monotonically decreasing and negative on $(0, +\infty)$, and consequently, we have

$$-\frac{3}{5} = \lim_{t \rightarrow +\infty} \varphi(t) < \varphi(t) \leq 0, \quad \forall t > 0.$$

This implies that

$$0 \leq |\varphi(v_i)| = -\varphi(v_i) < \frac{3}{5}, \quad \forall i = 1, \dots, n.$$

Then as $\|p_v\|^2 = \frac{4}{25}\delta^2$, $\|\varphi(v)\|_\infty = \max_i |\varphi(v_i)| < \frac{3}{5}$, and $\|p_v^2\| \leq \|p_v\|^2$, it yields

$$\left\| e - \frac{3}{5}v^2 - \frac{2}{5}v^{-3} \right\| \leq \|\varphi(v)\|_\infty \frac{25}{4}\|p_v\|^2 = \frac{3}{5}\delta^2.$$

Due to (12), $\|d_x d_y\| \leq \frac{\sqrt{2}}{25}\delta^2$, we get $\|e - v_+^2\| \leq \left(\frac{3}{5} + \frac{\sqrt{2}}{25}\right)\delta^2$. This implies that

$$\delta^+ \leq \left(\frac{5^2}{25 - \delta^2} + \frac{5^4}{(25 - \delta^2)^2} + \frac{5}{5 + \sqrt{25 - \delta^2}} \right) \left(\frac{3}{5} + \frac{\sqrt{2}}{25} \right) \delta^2.$$

Next, assume $\delta \leq \frac{1}{4}$, so $\delta^+ \leq 1.6466\delta^2$. This completes the proof. $\square$

The next lemma shows the influence of a full-Newton step on the duality gap and gives an upper bound for it.

Lemma 4.5. *Let $\delta = \delta(xy; \mu)$ and suppose that the vectors x_+ and y_+ are obtained using a full-Newton step, thus $x_+ = x + \Delta x$ and $y_+ = y + \Delta y$. We have $(x_+)^T y_+ \leq \mu(n + 2\delta^2)$.*

In addition, if $\delta \leq \frac{1}{4}$, then

$$(x_+)^T y_+ \leq 2\mu n. \quad (16)$$

Proof. Due to (6) and (8) we have

$$v^2 + vp_v + d_x d_y = \frac{3}{5}v^2 + \frac{2}{5}v^{-3} + d_x d_y,$$

then

$$\begin{aligned} x_+ y_+ &= \mu(v^2 + vp_v + d_x d_y) = \mu \left(e + \frac{25}{4}p_v^2 \cdot \frac{\frac{3}{5}v^2 + \frac{2}{5}v^{-3} - e}{(v^{-4} - v)^2} + d_x d_y \right) \\ &\leq \mu \left(e + \frac{25}{4}p_v^2 + d_x d_y \right), \end{aligned}$$

since after some reductions,

$$\frac{\frac{3}{5}v_i^2 + \frac{2}{5}v_i^{-3} - 1}{(v_i^{-4} - v_i)^2} = -\varphi(v_i) < \frac{3}{5} < 1, \quad \text{for all } i$$

where φ is defined in (15). Then from (10) and (11), we get

$$(x_+)^T y_+ = e^T (x_+ y_+) \leq \mu \left(n + \frac{25}{4} \|p_v\|^2 + \frac{2}{25} \delta^2 \right) \leq \mu(n + 2\delta^2).$$

Next, let $\delta \leq \frac{1}{4}$ so $\delta^2 \leq 1$. Using this fact we deduce that $(x_+)^T y_+ \leq \mu(n + 2)$. But since $n + 2 \leq 2n$, $\forall n \geq 2$. This completes the proof. $\square$

In the next theorem we analyze the effect of a full-Newton step on the proximity by updating the parameter μ by a factor $(1 - \theta)$.

Theorem 4.1. Let $\delta \leq \frac{1}{4}$ and $\mu_+ = (1 - \theta)\mu$, where $0 < \theta < 1$. Then

$$\delta(x_+ y_+; \mu_+) \leq \frac{5\sqrt{2n}\theta}{\sqrt{1-\theta}} + \delta^+.$$

In addition, if $\theta = \frac{1}{35\sqrt{2n}}$, $n \geq 2$, then $\delta(x_+ y_+; \mu_+) \leq \frac{1}{4}$.

Proof. As $\sqrt{\frac{x_+ y_+}{\mu_+}} = \frac{1}{\sqrt{1-\theta}} v_+$ and $\delta_+ = \|v_+^{-4} - v_+\|$, we have

$$\begin{aligned} \delta(x_+ y_+; \mu_+) &= \left\| \left(\sqrt{\frac{x_+ y_+}{(1-\theta)\mu}} \right)^{-4} - \sqrt{\frac{x_+ y_+}{(1-\theta)\mu}} \right\| = \left\| (1-\theta)^2 v_+^{-4} - \frac{1}{\sqrt{1-\theta}} v_+ \right\| \\ &= \left\| (1-\theta)^2 v_+^{-4} - (1-\theta)^2 v_+ + (1-\theta)^2 v_+ - \frac{1}{\sqrt{1-\theta}} v_+ \right\|. \end{aligned}$$

Now, from the triangular inequality it follows that

$$\begin{aligned} \delta(x_+ y_+; \mu_+) &\leq \left\| (1-\theta)^2 v_+^{-4} - (1-\theta)^2 v_+ \right\| + \left\| (1-\theta)^2 v_+ - \frac{1}{\sqrt{1-\theta}} v_+ \right\| \\ &= (1-\theta)^2 \delta^+ + \left| (1-\theta)^2 - \frac{1}{\sqrt{1-\theta}} \right| \|v_+\| \leq \delta^+ + \left| (1-\theta)^2 - \frac{1}{\sqrt{1-\theta}} \right| \|v_+\| \\ &= \delta^+ + \left| (1-\theta)^2 - \frac{\sqrt{1-\theta}}{1-\theta} \right| \|v_+\| = \delta^+ + \frac{1}{|1-\theta|} \left| (1-\theta)^3 - \sqrt{1-\theta} \right| \|v_+\| \\ &= \delta^+ + \left| \frac{(1-\theta)^5 - 1}{(1-\theta)^3 + \sqrt{1-\theta}} \right| \|v_+\| = \delta^+ + \left| \frac{-\theta(\theta^4 - 5\theta^3 + 10\theta^2 - 10\theta + 5)}{(1-\theta)^3 + \sqrt{1-\theta}} \right| \|v_+\|. \end{aligned}$$

Next, as $\frac{1}{(1-\theta)^3 + \sqrt{1-\theta}} \leq \frac{1}{\sqrt{1-\theta}}$, and $0 < \theta^4 - 5\theta^3 + 10\theta^2 - 10\theta + 5 \leq 5$, $\forall \theta \in (0, 1)$, and also due to (16) in Lemma 4.5, we have $\|v_+\| \leq \sqrt{2n}$, and thus it follows

$$\delta(x_+y_+; \mu_+) \leq \delta^+ + \frac{5\sqrt{2n}\theta}{\sqrt{1-\theta}}, \quad \forall \theta \in (0, 1).$$

By Lemma 4.4, $\delta^+ \leq 1.6466\delta^2$ and since $\delta \leq \frac{1}{4}$, then we get

$$\delta(x_+y_+; \mu_+) \leq 0.1029 + \frac{5\sqrt{2n}\theta}{\sqrt{1-\theta}}.$$

Let $\theta = \frac{1}{35\sqrt{2n}}$, $n \geq 2$, so $\theta \in [0, \frac{1}{70}]$ and thus we conclude that $\delta(x_+y_+; \mu_+) \leq h(\theta)$ where

$$h(\theta) = \frac{1}{7\sqrt{1-\theta}} + 0.1029.$$

As $h'(\theta) > 0$, $\forall \theta \in [0, \frac{1}{70}]$, hence $h(\theta)$ is an increasing function so $h(\theta) \leq h(\frac{1}{70}) = 0.2468$. Finally, we deduce that $\delta(x_+y_+; \mu_+) \leq \frac{1}{4}$. This proves the theorem. $\square$

Theorem 4.1 shows that Algorithm 3.1 is well defined, since $(x, y) \in \mathcal{N}$ is maintained throughout the algorithm.

Next, by regrouping the previous results, we end this section by stating the iteration bound of Algorithm 3.1.

Lemma 4.6. *Assume that the pair (x^0, y^0) is strictly feasible point and $\delta(x^0y^0; \mu_0) \leq \frac{1}{4}$ for a fixed $\mu_0 > 0$. Let x^k and y^k be the vectors obtained after k iterations with $\mu := \mu^k$. Then the inequality $(x^k)^T y^k \leq \epsilon$ is satisfied if $k \geq \left\lceil \frac{1}{\theta} \log \frac{n\mu_0}{\epsilon} \right\rceil$.*

Proof. We have $\mu^k = (1-\theta)\mu^{k-1} = (1-\theta)^k\mu_0$, then from Lemma 4.5 (16), $(x^k)^T y^k \leq (1-\theta)^k 2n\mu_0$. So $(x^k)^T y^k \leq \epsilon$ holds if $(1-\theta)^k 2n\mu_0 \leq \epsilon$. By taking the logarithm, we deduce that $k \log(1-\theta) \leq \log \epsilon - \log 2n\mu_0$. Next, by the inequality $-\log(1-\theta) \geq \theta$ for all $0 < \theta < 1$, we obtain $k\theta \geq \log 2n\mu_0 - \log \epsilon = \log \frac{2n\mu_0}{\epsilon}$. This proves the lemma. $\square$

Theorem 4.2. *Using defaults $\theta = \frac{1}{35\sqrt{2n}}$, $n \geq 2$ and $\delta = \frac{1}{4}$ where $\mu_0 = \frac{1}{2}$. Algorithm 3.1 requires at most $\mathcal{O}(\sqrt{n} \log \frac{n}{\epsilon})$ iterations for getting an ϵ -approximate solution of monotone LCP in (1).*

Proof. Let $\theta = \frac{1}{35\sqrt{2n}}$ and $\mu_0 = \frac{1}{2}$, by using Lemma 4.6, the proof is straightforward. $\square$

5. NUMERICAL RESULTS

To evaluate the performance of Algorithm 3.1, we consider some examples of monotone LCP problems of different sizes, each example is followed by two tables one contains initial starting points and the other summarizes numerical results obtained by the algorithm. We implement Algorithm 3.1, on the software *MATLAB 7.9* and run on a *PC* with *CPU 2.13 GHz* and *2G RAM* memory and double precision format on some examples of monotone LCP. In our implementation, we use $\epsilon = 10^{-4}$ and $\mu_0 = \frac{(x^0)^T y^0}{n}$. In addition, we compare the obtained numerical results by Algorithm 3.1 with those obtained by Algorithm in Figure 1 (*Kh-N*) [29], where their proposed threshold τ and update barrier parameter θ if $q = 5$ are given by $\left(\tau = \frac{1}{9}, \theta = \frac{1}{704\sqrt{n}}\right)$. A solution of the monotone LCP is denoted by x^* . Finally, the “Iter” and “CPU” denote the number of iterations and the elapsed time in seconds, respectively.

5.1. Numerical results with the original version of Algorithm 3.1

In this subsection, we present our numerical results obtained by our Algorithm 3.1 and those given by Kheirfam and Nasrollahi [29].

Problem 1. Consider the following monotone LCP where $M \in \mathbb{R}^{5 \times 5}$ and $q \in \mathbb{R}^5$ are given by:

$$M = \begin{pmatrix} 6 & 6 & 4 & 3 & 2 \\ 8 & 21 & 14 & 10 & 12 \\ 4 & 14 & 13 & 5 & 9 \\ 4 & 10 & 5 & 6 & 5 \\ 3 & 12 & 8 & 4 & 10 \end{pmatrix}, \quad q = \begin{pmatrix} -20.5 \\ -64.5 \\ -44.5 \\ -29.5 \\ -36.5 \end{pmatrix}.$$

For this example, the initial point for the algorithm is taken as

$$x^0 = (1, 1, 1, 1, 1)^T, \quad y^0 = Mx^0 + q = (0.5, 0.5, 0.5, 0.5, 0.5)^T,$$

with the proximity $\delta(x^0, y^0, \mu_0) = 0$, *i.e.*, $(x^0, y^0) \in \mathcal{N}(\tau, \mu_0)$.

A solution of Problem 1 is:

$$x^* = (0.6364, 2.3222, 0.5847, 0, 0.2046)^T, \quad y^* = (0, 0, 0, 0.2149, 0)^T.$$

Problem 2. In this example of monotone LCP, $M \in \mathbb{R}^{8 \times 8}$ and $q \in \mathbb{R}^8$ are given by:

$$M = \begin{pmatrix} 8 & 9 & 13 & 13 & 5 & 11 & 9 & 10 \\ 8 & 10 & 15 & 15 & 7 & 12 & 10 & 12 \\ 13 & 15 & 26 & 26 & 10 & 20 & 13 & 21 \\ 13 & 15 & 26 & 26 & 10 & 20 & 12 & 20 \\ 5 & 7 & 10 & 10 & 5 & 9 & 5 & 8 \\ 11 & 12 & 20 & 20 & 9 & 19 & 13 & 15 \\ 9 & 10 & 13 & 12 & 5 & 13 & 16 & 13 \\ 10 & 12 & 21 & 20 & 8 & 15 & 13 & 22 \end{pmatrix}, \quad q = \begin{pmatrix} -8.265 \\ -9.3033 \\ -14.835 \\ -14.4633 \\ -5.995 \\ -12.4133 \\ -10.015 \\ -12.3033 \end{pmatrix}.$$

The initial starting points considered for Problem 2 is

$$x^0 = (0.2233, 0.1893, 0.1207, 0.1202, 0.2758, 0.1431, 0.1961, 0.1403)^T, \\ y^0 = Mx^0 + q = (4.4778, 5.2820, 8.2864, 8.3217, 3.6254, 6.9875, 5.1000, 7.1288)^T,$$

with the proximity $\delta(x^0, y^0, \mu_0) = 0.0011$, *i.e.*, $(x^0, y^0) \in \mathcal{N}(\tau, \mu_0)$.

A solution of Problem 2 is:

$$x^* = (0.1948, 0, 0.2655, 0, 0.2506, 0, 0.2221, 0)^T, \\ y^* = (0, 0.2165, 0, 0.1495, 0, 0.1872, 0, 0.1178)^T.$$

Problem 3. Let us consider the monotone LCP where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$ are given by:

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{pmatrix}, \quad q = -Me + e.$$

For this example, we consider the following initial starting point $x^0 = y^0 = e$, where the proximity $\delta(x^0, y^0, \mu_0) = 0$, *i.e.*, $(x^0, y^0) \in \mathcal{N}(\tau, \mu_0)$.

TABLE 1. Number of iterations and CPU time for Problems 1–3.

	$n \downarrow$	Algorithm 3.1		Algorithm Kh-N	
		ITER	CPU	ITER	CPU
Problem 1	5	1116	0.5311	15937	4.9717
Problem 2	8	1479	0.8817	21095	7.7729
	5	1193	0.5481	17027	5.1758
	10	1797	1.1259	25625	12.7354
Problem 3	20	2696	2.4844	38424	31.6842
	30	3413	4.4300	48624	70.1209
	50	4587	13.6326	–	–
	100	6832	120.4609	–	–

In Table 1, the obtained numerical results for Problems 1–3 are summarized.

Comment 1. Across the Table 1, we see that the number of iterations and the elapsed time produced by our algorithm by using our theoretical default $\theta = \frac{1}{35\sqrt{2n}}$ are too much higher. This means that the algorithm converges to a solution of LCP but very slowly. Meanwhile, for the default $\theta = \frac{1}{704\sqrt{n}}$ according to [29] with $q = 5$, the algorithm diverges when n reaches the size $n \geq 50$. It is clear that our numerical results are better than those given by the algorithm in [29].

In general, the obtained numerical results by using both θ are not good. The cause is due to the fact that the defaults of θ become very small for large size problems. This leads during the execution of algorithms that the rate of decrease $(1 - \theta)$ in the sequence of barrier parameters $\{\mu_k\}$ becomes close enough to one.

5.2. A numerical amelioration for the algorithm

In this subsection, in order to ameliorate the performance of *Algorithm 3.1*, we import some changes, where instead of using the theoretical default of θ , we relaxed them to a constant which belongs to $(0, 1)$ and we use $\epsilon = 10^{-7}$. Further, to keep the iterates positive, we introduce at each iteration k , a step-size $\alpha^k > 0$ such that $x^k + \rho\alpha^k(\Delta x)^k > 0$ and $y^k + \rho\alpha^k(\Delta y)^k > 0$ with $\alpha^k = \min\{1, \bar{\alpha}\}$ and $\rho \in (0, 1)$ where $\bar{\alpha} = \min\{\alpha_x, \alpha_y\}$ and α_x, α_y are given by

$$\alpha_x = \{-(x_i)^k/(\Delta x_i)^k : (\Delta x_i)^k < 0\}; \quad \alpha_y = \{-(y_i)^k/(\Delta y_i)^k : (\Delta y_i)^k < 0\}$$

Our obtained numerical results based on those modifications are compared with Fischer’s Algorithm [20] and summarized in Table 2.

Comment 2. Across the Table 2, we see that the number of iterations and the elapsed time produced by the ameliorated algorithm when using the default θ as a constant are too much lower. Meanwhile, for the algorithm proposed by [20], the algorithm diverges when n reaches the size $n > 100$. It is clear that our numerical results are better than those given by the algorithm in [20]. This due to the quadratic convergence of our algorithm (see Lemma 4.4).

The numerical results presented in Table 1, while confirming the theory, are done on some very simple and artificial examples, we decided to present in the second part of this section the results obtained by running our algorithm on some selection of *LCPs* from the *NETLIB* repository [21], the obtained numerical results for this set of problems are summarized in Table 3. In this case for the choice of feasible initial points we used the homogeneous and self-dual LCPs model in [38].

TABLE 2. Number of iterations and CPU time for Problems 1–3.

		Ameliorated algorithm 2.3				Fischer's algorithm	
		$\theta = 0.7$		$\theta = 0.9$			
	$n \downarrow$	ITER	CPU	ITER	CPU	ITER	CPU
Problem 1	5	11	0.0070	6	0.0043	21	0.9959
Problem 2	8	11	0.0084	6	0.0055	24	1.86589
	10	11	0.0092	6	0.0061	24	2.1962
	20	12	0.0150	6	0.0076	26	3.1721
Problem 3	50	13	0.0626	7	0.0364	29	5.1747
	100	13	0.3405	7	0.2154	31	7.1820
	500	15	6.0800	8	3.9539	–	–
	1000	16	44.6532	8	22.9708	–	–

TABLE 3. Number of iterations and CPU time for NETLIB set problems.

		Ameliorated algorithm			
		$\theta = 0.55$		$\theta = 0.65$	
Problem	$n \downarrow$	ITER	CPU	ITER	CPU
afiro	78	26	0.5634	20	0.4593
kb2	111	27	1.2623	20	0.9382
sc50b	128	27	1.5837	20	1.1549
blend	188	27	3.7828	21	3.1589
adlittle	194	27	3.5741	21	2.7790
share2b	258	28	5.9648	21	5.3182
stocfor1	282	28	11.9502	21	8.2487
recipe	295	28	6.6336	21	5.0846
scagr7	314	28	11.7109	21	8.8844
share1b	370	28	19.5827	21	23.6124
grow7	441	28	39.6760	22	26.6245
beaconfd	468	28	31.4545	22	27.4596
e226	695	29	114.0905	22	68.2650
capri	753	29	115.2951	22	96.8791
bandm	777	29	174.6404	22	138.2905
agg	1103	30	329.4844	24	212.2758

6. CONCLUSION AND REMARKS

In this paper, we have presented a feasible full-Newton step IPA for monotone LCPs. For this purpose, we have introduced a function ψ to the equation which characterizes the central-path of LCPs and we have applied Newton's method to offer new search directions. For $\psi(t) = t^{\frac{5}{2}}$, we have proved its polynomial complexity under appropriate defaults of τ and θ . The evaluation of Algorithm 3.1, across the obtained numerical results, showed its slow convergence and even for the version proposed by [29]. We note also that Fischer's algorithm failed for monotone LCPs with size $n > 100$. However, the import ameliorations made on our algorithm, the new obtained numerical results on NETLIB collection of monotone LCP problems, are significantly improved. Finally, an interesting topic of research in the future is the extension of Algorithm 3.1 to the large class of $P_*(\kappa)$ -LCPs. Moreover, the development of an infeasible full-Newton step IPA for monotone LCP based on our AET remains a good subject of research.

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REFERENCES

- [1] M. Achache, A weighted-path-following method for the linear complementarity problem. *Studia Universitatis Babeş-Bolyai. Series Informatica* **49** (2004) 61–73.
- [2] M. Achache, A new primal-dual path-following method for convex quadratic programming. *Comput. Appl. Math.* **25** (2006) 97–110.
- [3] M. Achache, Complexity analysis and numerical implementation of a short-step primal-dual algorithm for linear complementarity problems. *Appl. Math. Comput.* **216** (2010) 1889–1895.
- [4] M. Achache, Complexity analysis of an interior point algorithm for the semidefinite optimization based on a kernel function with a double barrier term. *Acta Math. Sin.* **31** (2015) 543–556.
- [5] M. Achache, A new parameterized kernel function for LO yielding the best known iteration bound for a large-update interior point algorithm. *Afr. Mat.* **27** (2016) 591–601.
- [6] M. Achache and R. Khebbache, A full-Newton step feasible weighted primal-dual interior point algorithm for monotone LCP. *Afr. Mat.* **26** (2015) 139–151.
- [7] M. Achache and N. Tabchouche, A full-Newton step feasible interior-point algorithm for monotone horizontal linear complementarity problems. *Optimiz. Lett.* **13** (2018) 1039–1057.
- [8] M. Achache and N. Tabchouche, Complexity analysis and numerical implementation of large-update interior-point methods for SDLCP based on a new parametric barrier kernel function, Optimization. *J. Math. Prog. Oper. Res.* **67** (2018) 1211–1230.
- [9] Y.Q. Bai, M. El Ghami and C. Roos, A new efficient large-update primal-dual interior-point method based on finite barrier. *SIAM J. Optimiz.* **13** (2002) 766–782.
- [10] Y.Q. Bai, M. El Ghami and C. Roos, A comparative study of kernel function for primal-dual interior-points algorithms in linear optimization. *SIAM J. Optimiz.* **15** (2004) 101–128.
- [11] R.W. Cottle, J.S. Pang and R.E. Stone, *The Linear Complementarity Problem*, Academic, San Diego (1992).
- [12] J. Czyzyk, S. Mehrotra, M. Wagner and S.J. Wright, PCx: an interior-point code for linear programming. *Optim. Methods Softw.* **11** (1999) 397–430.
- [13] Z. Darvay, A new algorithm for solving self-dual linear optimization problems. *Stud. Univ. Babeş-Bolyai Inform.* **47** (2002) 15–26.
- [14] Z. Darvay, New interior point algorithm in linear programming. *Adv. Model. Optimiz.* **5** (2003) 51–92.
- [15] Z. Darvay and P.R. Takács, New method for determining search directions for interior point algorithms in linear optimization. *Optimiz. Lett.* **12** (2018) 1099–1116.
- [16] Z. Darvay, I.M. Papp and P.R. Takács, Complexity analysis of a full-Newton step interior point method for linear optimization. *Period. Math. Hung.* **73** (2016) 27–42.
- [17] Z. Darvay, T. Illés and C. Majoros, Interior-point algorithm for sufficient LCPs based on the technique of algebraically equivalent transformation. *Optimiz. Lett.* **15** (2021) 357–376.
- [18] Z. Darvay, T. Illés, J. Povh and P.R. Rigó, Feasible corrector-predictor interior-point algorithm for $P_*(\kappa)$ -linear complementarity problems based on a new search directions. *SIAM J. Optimiz.* **30** (2020) 2628–2658.
- [19] A. Fischer, A special Newton-type optimization method. *Optimization* **24** (1992) 269–284.
- [20] A. Fischer, On the local superlinear convergence of a Newton-type method for LCP under weak conditions. *Optimiz. Methods Softw.* **6** (1995) 83–107.
- [21] D. Gay, Electronic mail distribution of linear programming test problems. *Math. Program. Soc. COAL News* **13** (1985) 10–12.
- [22] W. Grimes, Path-following interior-point algorithm for monotone linear complementarity problems. *Asian-Eur. J. Math.* **15** (2022).
- [23] W. Grimes and M. Achache, An infeasible interior-point algorithm for monotone linear complementarity problems. *Int. J. Inf. Appl. Math.* **04** (2021) 53–59.
- [24] M. Haddou, T. Migot and J. Omer, A generalized direction in interior point method for monotone linear complementarity problems. *Optimiz. Lett.* **13** (2019) 35–53.
- [25] N.K. Karmarkar, A new polynomial-time algorithm for linear programming. *Combinatorica* **4** (1984) 373–395.
- [26] B. Kheirfam, A predictor-corrector interior-point algorithm for $P_*(k)$ -horizontal linear complementarity problem. *Numer. Algorithms* **66** (2014) 349–361.
- [27] B. Kheirfam, A new search direction for full-Newton step infeasible interior-point method in linear optimization. Preprint [arXiv:2102.07223v1](https://arxiv.org/abs/2102.07223v1) (2021).
- [28] B. Kheirfam and M. Haghighi, A full-Newton step feasible interior-point algorithm for $P_*(k)$ -LCP based on a new search direction. *Croat. Oper. Rev.* **7** (2016) 277–290.
- [29] B. Kheirfam and A. Nasrollahi, A full-Newton step interior-point method based on a class of specific algebra transformation. *Fundam. Inf.* **163** (2018) 325–337.
- [30] M. Kojima, N. Megiddo, T. Noma, A. Yoshise, A unified approach to interior point algorithms for linear complementarity problems, in *Lecture Notes in Computer Science*, vol. 538. Springer-Verlag, Berlin (1991).

- [31] N. Moussaoui and M. Achache, A weighted-path following interior-point algorithm for convex quadratic optimization based on modified search directions. *Stat. Optimiz. Inf. Comput.* **10** (2022) 873–889.
- [32] S. Pan, X. Li and S. He, An infeasible primal-dual interior point algorithm for linear programs based on logarithmic equivalent transformation. *J. Math. Anal. Appl.* **314** (2006) 644–660.
- [33] J. Peng, C. Roos and T. Terlaky, Self-Regular Functions: A New Paradigm for primal-dual Interior-Point Methods, Princeton University Press, Princeton (2002).
- [34] C. Roos, T. Terlaky and J.P. Vial, Theory and algorithms for Linear Optimization. An Interior Point Approach, John-Wiley. Sons, Chichester, UK (1997).
- [35] G.Q. Wang and Y.Q. Bai, A new primal-dual path-following interior point algorithm for semidefinite programming. *J. Math. Anal. Appl.* **353** (2009) 339–349.
- [36] G.Q. Wang and Y.Q. Bai, A primal-dual interior-point algorithm for second-order cone optimization with full-Nesterov-Todd step. *Appl. Math. Comput.* **215** (2009) 1047–1061.
- [37] G.Q. Wang, X.J. Fan, D.T. Zhu and D.Z. Wang, New complexity analysis of a full-Newton step feasible interior-point algorithm for $P_*(k)$ -LCP. *Optimiz. Lett.* **9** (2015) 1105–1119.
- [38] Y. Ye, On homogeneous and self-dual algorithm for LCP. *Math. Prog.* **76** (1997) 211–222.
- [39] Y. Ye, Interior Point Algorithm, Theory and Analysis, John Wiley and Sons, Chichester, UK (1997).
- [40] Y. Ye, M.J. Todd and S. Mizuno, An $O(\sqrt{n}L)$ -iteration homogeneous and self-dual linear programming algorithm. *Math. Oper. Res.* **19** (1994) 53–67.
- [41] L. Zhang and Y. Xu, A full-Newton step primal dual interior-point algorithm for linear complementarity problem. *J. Inf. Comput. Sci.* **5** (2010) 305–313.



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