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AN INTEGRATED INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEM UNDER CREDIT POLICY AND PARTIAL BACKLOGGING WITH ADVERTISING AND PRICE DEPENDENT STOCHASTIC DEMAND

Sangita Mandal, Ateka Banu and Shyamal Kumar Mondal*

Abstract. In today's business world, advertising is one of the most important policies to attract more customers. This policy increases the retailer's sales and makes the retailer's business position strong. In this paper, we have considered an integrated inventory model for non-instantaneous deteriorating items with a single supplier and single retailer, where a supplier sells his/her products in the market through a retailer who faces a stochastic demand depending on both retail price and advertisement. Here, to increase the retailer's demand, the supplier wholesales his/her products to the retailer with a credit period. Since the long credit period increases the demand rate but at the same time, it increases the supplier's opportunity loss. In this paper, we have discussed about the credit policy and find out, how this policy effective on profit of the supplier, the retailer as well as the whole system. After purchasing, the retailer sells his/her products to his customers with a markup. This markup is based on the supplier's wholesale price and also on advertising costs. Finally, an integrated profit function has been developed and we have illustrated numerical examples to justify the feasibility of the proposed model. The result indicates that not only the credit period but also an appropriate number of advertisements is more important for maintaining the profit of the supplier, the retailer as well as the integrated profit of the system. We have analyzed the effect of markup on the profit function. We have also analyzed how the profit structure of the suppliers and retailers changes along with the changes in the length of the credit period. An effective algorithm has been presented in the solution procedure to find the optimal solutions of the proposed model. Also, the numerical example with uniform distribution has been carried out. Finally, sensitivity analysis of major parameters has been illustrated to provide managerial insights

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1. Introduction

Inventory is a stock of goods that a seller store and sells to his/her customers. His/her sales rate depends on the customer's demand. So, sellers use different ways to increase customer demand to increase their sales

Keywords. Integrated policy, Credit period, Advertising, Stochastic demand, Partial backlogging.

Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102, W.B., India.
*Corresponding author: shyamal@mail.vidyasagar.ac.in

rate. One of them is advertising. Previously many researchers such as Shah and Pandey [51], Das *et al.* [6], Manna *et al.* [28]. Khara *et al.* [22] works on advertising policy. Geetha and Udayakumar [12] developed a model for non-instantaneous deteriorating item with advertisement and price dependent demand. Through this advertisement, not only does the seller make his/her product attractive to the customers, but as a result, the customers get to know a lot more information about the item and their interest in buying the item increases. As a result of advertising, such as customer demand increases, but the continuous advertising of the product makes a customer monotonous. So, the customer's demand may be decreased. In the present competitive market, supply chain management plays an important role in the economy. We know the Covid-19 epidemic has a devastating effect on the world economy. The International Monetary Fund (IMF) described the situation as a new recession and compared it to the 2009 European financial crisis. A study released by the Asian Development Bank (ADB) estimates that the potential for economic damage from an epidemic is estimated at 77 billion to 347 billion dollars, which is 0.1–0.4% of GDP. Also, many people have lost their job due to this Covid-19. As a result of this financial crisis, people are always interested in buying low priced items. So, if the price of the goods is too high, then it has a negative impact on demand. Besides, in reality, it is not always possible to accurately determine the demand for the product, no matter how many advertisements are actually given. *Noting this aspect of reality, here we have considered advertising and price dependent stochastic demand function.*

This model assumes many assumptions which apply to real-life applications. Many researchers directed their study by assuming instantaneous deterioration, *i.e.*, the products started to deteriorate instantly. But not all products follow this instantaneous deterioration. Ouyang *et al.* [33] identified this fact as "non-instantaneous deterioration (NID)". In reality, there are many products, such as fruits, vegetables, meat products, foods etc. which are started to deteriorate after a certain time due to several reasons, such as spoilage, dryness and evaporation. That means this type of product remains fresh for a certain period of time. This situation is called "non-instantaneous deterioration". *We have considered the model for non-instantaneous deteriorating items.*

In reality, the supplier or the retailer's main aim is to attract more customers, increase their profit and reduce inventory level. To attract the retailer, the supplier gives offers of delayed payment (credit period). All the big companies are based on credit periods. All the construction industry is notorious for credit policy. A strong credit policy is one of the many tools that construction companies use to speed up payment, maintain a positive bank balance and even take on bigger projects. A credit policy is actually an incredibly helpful communication tool for building strong customer relationships in construction. In reality, the suppliers of the TATA steel company offer a credit period to their retailer. The two forms of credit policy: (a) full credit (b) partial credit. In a full credit supplier allows his/her customer to payment of all purchasing costs after the allowed time. But in that case, the supplier faced a huge amount of opportunity loss in that credit period. Also, in a partial credit, customers need to pay a portion of the total purchase cost at the ordering time and the rest to be paid during the allowed time. In previous works many researchers such as, Manna *et al.* [27], Banu and Mondal [3], Panja and Mondal [39] developed an inventory model with full credit policy but Hung and Hsu [17], Tiwari *et al.* [47] considered partial credit policy. *In this model, we have discussed partial credit policy.*

Also, markup plays an important role in the profitability of any business. On the one hand, if the markup is too high, the customer's interest in buying the product decreases, while if the markup is too low, the retailer's profit is adversely affected. So, the retailer should always markup keeping these two aspects in mind. Previously many researchers considered the markup only depends on purchase price. *We are the first to consider that the retailer sells his/her products to the customers with a markup over the purchase price, where the markup does not depend only on the purchase price but also on the number of advertisements given by him/her.*

The seller maintains the stability of his/her business by meeting the demands of his/her customers in the market. But sometimes customers cannot meet their demands due to a shortage of products. At that time some customers wait for backorders to satisfy their demand from the same retailer but some switch to another retailer to fulfilled their requirements. These two different situations for shortages (unsatisfied demand) are called respectively (a) partially backlogged (b) lost sales. In this regard, when all customers wait for backorders, called "full backlogged".

The main objectives of this paper are to find answers to the following questions:

(i) For uncertain demand, we have to calculate the retailer's optimum ordering quantity.
(ii) How does the credit policy effective on the retailer's ordering quantity? We know that the credit policy is beneficial for a retailer, can the retailer gradually increase ordering quantity with increasing credit period?
(iii) In the context of profit, how beneficial is credit policy?
(iv) What is the effect of advertisement frequency on the profit of the whole system?
(v) What is the optimum markup of the retailer for the optimum demand rate?

2. Literature review

2.1. Inventory model with stochastic demand

Gupta and Wang [14] expand a stochastic inventory model with trade credit. AlDUrgam *et al.* [1] consider a single vendor, single manufacturer integrated inventory model with stochastic demand. Kaur [20] introduced stochastic demand with two level trade credit with default risk in supply chain. Jauhari and Wangsa [18] developed an inventory model with remanufacturing, stochastic demand and green investments. Sicilia *et al.* [50] studied an optimal policy for multi-item systems by considering the stochastic demands and shortages where the shortages are backlogged.

2.2. Inventory model with credit policy

Credit policy gives advantages to the supplier as well as retailer: it attracts new retailer for there purchases, which will help to achieve its sales target. Also, for those who want to start a business but do not have sufficient money, the credit policy helps them to start a business. Mehta [30] first formulated the formula of credit period. The first Haley and Higgins [15] first developed the inventory model with trade credit policy. Luo [26] developed an inventory model of Buyer-Vendor inventory coordination with credit period incentives. Huang and Hsu [17] developed an EOQ model under retailer partial trade credit policy. Mishra *et al.* [31] considered a trade credit and price dependent demand in maximizing the total profit of the retailer with a controllable deterioration rate. Manna *et al.* [27] developed a two layer green supply chain for imperfect production inventory model under bi-level credit period. Tavakoli and Taleizadeh [41] developed an EOQ model for decaying item by considering full advanced payment and conditional discount. Tiwary *et al.* [46] consider an inventory model for deteriorating items under two level partial trade credit policy. Johari *et al.* [19] consider a inventory system with bi-level credit period coordination and price credit dependent demand. Panja and Mondal [38] analysing a four layer green supply chain inventory model under type-2 fuzzy credit period. Banu and Mondal [3] considered an inventory model with two-level trade credit period using q-fuzzy number. Panja and Mondal [39] developed a two layer green supply chain with credit linked demand and mark up under revenue sharing contract. Panja and Mondal [40] considered an imperfect four-layer production inventory model under two-level credit period using branch-and-bound technique. Mashud *et al.* [29] considered a resilient hybrid payment supply chain inventory model. Khara *et al.* [21] developed an imperfect production inventory model with advance payment and credit period. Banu *et al.* [4] considered a supply chain model with adjustment of credit period and stock dependent demands. Taleizadeh *et al.* [45] studied an inventory model by considering partial linked-to-order delayed payment and life time effects on decaying items ordering. Tiwari *et al.* [47] developed a model for imperfect quality and deteriorating items under two-level trade credit.

2.3. Inventory model for deteriorating items

In reality no product stays fresh in lifetime. They are all deteriorate, but not all products deteriorate immediately. They are started to deteriorate after a certain time and this products are called non-instantaneous deteriorating items. Many researchers introduced an inventory model for deteriorating items under numerous conditions. Taleizadeh *et al.* [44] developed an economic order quantity model for deteriorating items by considering quantity discount and prepayment. Taleizadeh and Nematollahi [43] consider an inventory control problem for deteriorating items with backordering. Taleizadeh [42] developed an EOQ model for deteriorating items by considering multiple prepayments. Lashgari *et al.* [23] developed an inventory control problem for deteriorating item with backordering. Diabat *et al.* [8] studied a model for deteriorating items by considering partial downstream delayed payment, partial upstream advance prepayment and partial backordering. Lashgari *et al.* [24] developed an ordering policies for non-instantaneous deteriorating items by considering hybrid partial prepayment, partial delay payment and partial backordering. Nobil *et al.* [32] consider single-machine lot scheduling problem for deteriorating items with negative exponential deterioration rate.

2.4. Inventory model with allowable shortages and backorder

Demand plays an important role in the profit of the supplier or retailer. If the demand rate is deterministic then the supplier can bring the items according to the demand of his customers and fulfil the customer's requirements, immediately. But, if the demand is stochastic then supplier has to bring the averages goods and most of the time the demand is not fulfilled by supplier immediately, thus a stock out situation occurs. As a results some customers go to an alternative store to fulfilled their requirements and some are wait for the next lot, is known as partial backordered. Gani and Maheswari [11] developed and EOQ model for imperfect quality item where shortages are backordered. Hsu and Hsu [16] developed an EPQ model for imperfect production process with backordered. Ganesh Kumar and Uthayakumar [10] developed an multi item inventory model with variable backordered. Zita [53] developed a fuzzy model with allowable shortages and backordered. San-Jose *et al.* [49] studied an inventory system by considering time-dependent demand and partial backordering. Palanivel and Suganya [36] developed an partial backlogging inventory model where the demand depend on price and stock level.

2.5. Inventory model with advertising policy

Advertising is one of the policies to increase business profit. As a result of advertising, the good aspects of the product are presented to the customers and as a result the product becomes very attractive to the customers. Das *et al.* [6] considered an integrated inventory model with delay payment for deteriorating item and advertising and price dependent demand. Palanivel and Uthayakumar [37] developed an EOQ model for non-instantaneous deteriorating items with price and advertisement dependent demand under inflation. Then Geetha and Udayakumar [12] considered advertisement and price dependent demand in their inventory model for non-instantaneous deteriorating items. Manna *et al.* [28] developed a imperfect production inventory model with production rate dependent defective rate and advertisement dependent demand. Rathore and Sharma [48] developed a preservation technology model for deteriorating items with advertisement dependent demand and partial trade credit. Khara *et al.* [22] considered an integrated imperfect production system with advertisement dependent demand.

2.6. Integrated system inventory model

In an integrated supply chain the supplier, retailer, manufacturer all were working together to increase the total profit of the whole supply chain. Many researchers developed integrated inventory model. Das *et al.* [5] developed an integrated supply chain model for deteriorating item with procurement cost dependent credit period. Das *et al.* [7] considered an integrated production inventory model with defective item with stochastic credit period. Banu *et al.* [2] considered an integrated inventory model with warranty dependent credit period. Lin *et al.* [25] developed an integrated inventory model with impacts of two-stage deterioration under trade credit and variable capacity utilization. Yadav *et al.* [52] considered an manufacturer-retailer integrated inventory model with controllable lead time and service level constraint. Padiyar *et al.* [34] developed an integrated inventory model for imperfect production process.

TABLE 1. Comparison table of previous inventory model with present model.

Authors	Credit policy	Credit dependent demand	Stochastic demand	Advertising	Deterioration	Shortages policy	Integrated
Giri [13]	–	–	–	–	–	✓	–
Huang and Hsu [17]	✓	–	–	–	–	–	–
Pal *et al.* [35]	–	–	✓	–	–	–	–
Geetha and Udayakumar [12]	–	–	–	✓	✓	✓	–
Lashgari *et al.* [24]	✓	–	–	–	✓	✓	–
Johari *et al.* [19]	–	–	✓	–	–	✓	–
Kaur [20]	✓	✓	✓	–	–	✓	✓
Zita [53]	–	–	–	–	–	✓	–
Mashud [29]	–	–	–	–	✓	✓	–
Present Model	✓	✓	✓	✓	✓	✓	✓

After reviewing the literature, a comparison table showing different inventory problems under various conditions have been represented in Table 1. This paper is a modification of previous studies by adaptation full and partial credit period, supplier demand, partial backlogging and integrated profit in a system with advertisement and price-dependent stochastic demand to maintain the stability of the market profits in the post-pandemic situation.

2.7. Research gap and contribution

The literature review of the paper illustrates above that the six topics have been studied in the past. The differences between our proposed model and the previous model are described as follows. First, some inventory model with stochastic demand has been developed to investigate the impact of stochastic demand on inventory decisions, but no one has studied the influence of incorporating price and advertising on the demand function. In previous works, Kaur [20], Jauhari and Wangsa [18], Sicilia *et al.* [50] mostly considered stochastic demand, where the whole demand is considered to be stochastic but in the proposed model, we considered an advertising and price-dependent demand function where the normal market demand is considered to be stochastic. Secondly, most of the previous works, such as Johari *et al.* [19], Panja and Mondal [38], Mashud *et al.* [29], Duary *et al.* [9] considered a model with only one type of credit period, which is full or partial, but in the proposed model, we have discussed the full and partial both credit periods and find out which is more profitable for the supplier, the retailer as well as the whole system. Also, in previous works, the supplier or retailer usually fixed the markup based on their purchase cost, but we are the first to considered that the markup is not only based on the purchase cost, it also depends on his given number of advertisements.

The rest of the paper is organised as follows. Section 3 describes the problem by explaining some notations which have been used in this paper and the required assumptions to develop the model. In Section 4, formulates the model. In Section 5, the solution procedure has been presented to solve the model. In Section 6, a numerical example is illustrated and the sensitivity analysis is dedicated. Section 7 describes managerial implications. Finally, conclusions and future research directions are presented in Section 8.

3. Problem description

In this study, we have considered a model with a single supplier and single retailer, where the system has an infinite planning horizon. In this model, we have discussed the non-instantaneous deteriorating items *i.e.*, those items which remain fresh for a certain period of time and then start to deteriorate. In most of the developing countries, the retailers or customers do not always have sufficient on hand cash to start a business, so in this case credit policy plays an important role. Here, the supplier sells his/her products to the retailer by giving a partial credit policy. In this policy, the retailer pays a portion of purchase cost to the supplier at the time of purchase and the remaining portion is paid over the credit period. The length of this credit period plays a significant role in the profit structure of both the supplier and the retailer. In this model, we discuss the profitability in

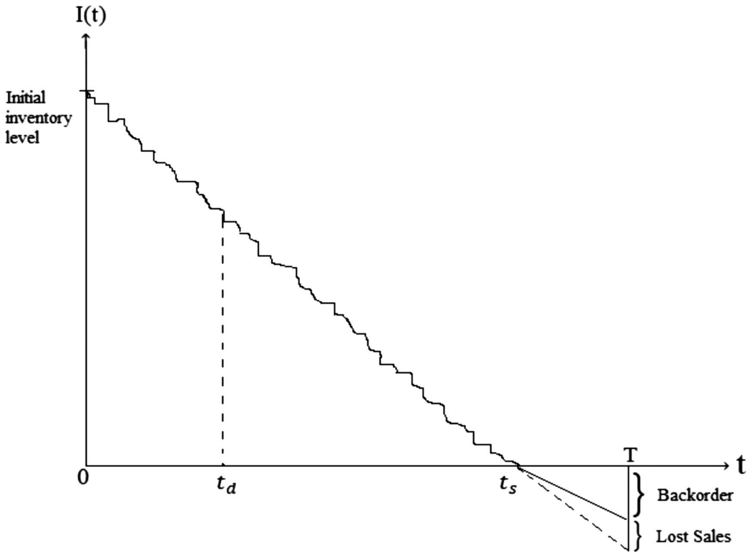


FIGURE 1. Retailer's inventory level.

terms of different lengths of the credit period for non-instantaneous deteriorating items, there may exists three cases depending on the credit period on the basis of the time of commencement of deterioration of the items and the time of shortages *i.e.*, if M be the credit period offered by the supplier and t_d, t_s and T be the length of time in which the product remains fresh, the length of the time of on hand inventory and the length of the inventory cycle respectively. Then the credit period M may belong to (1) $0 \leq M \leq t_d$, (2) $t_d \leq M \leq t_s$, (3) $t_s \leq M \leq T$. Here, the retailer's demand rate depends on the supplier's offered credit period. To attract more customers, the retailer gives advertisements about his/her product. Providing this advertisement, the retailer increases his/her customer's demand, although continuous advertisement has a negative impact on the demand. At the time of retailing, the retailer sells his/her product with a markup. This markup depends not only on the purchase price but also on the advertising frequency. With the increase in the number of advertisements, as the advertisement cost increases, the retailer increases the markup to maintain its profit. Also, more markup increases the retailer's retail price, which has a negative impact on the customer's demand. So, the retailer has to increase the markup along with the advertisement in such a way that it does not have any negative impact on his profit. Hence, it can be said that in this model, the advertising, pricing and markup are highly interrelated. Here, the normal market demand is a random variable which follows a probability density function. Also, we have considered shortages. So, the customers fail to meet their requirements, and some of them wait for the backorder. This unsatisfied demand is called partially backlogged.

3.1. Notations

For convenience, the following notations are used throughout the paper.

D	Demand for the product.
A	Ordering cost per order.
p	Supplier purchase cost per unit item.
c_p	Retailer's purchase cost per unit item.
c_r	Retail price per unit item.
c_a	Advertising cost per advertisement.
c_h	Holding cost of product unit per unit time.

c_b	Backorder cost.
c_l	Cost of lost sale per unit.
$I_1(t)$	Inventory level at any time t, where $0 \leq t \leq t_d$.
$I_2(t)$	Inventory level at any time t, where $t_d \leq t \leq t_s$.
$I_3(t)$	Inventory level at any time t, where $t_s \leq t \leq T$.
I_e	Interest rate earned by the retailer in a year.
I_c	Interest rate charged per year.
i_c	Banking interest rate per year.
I_l	Supplier's interest loss rate due to offered credit period.
δ	Fraction of purchase cost that will be paid at ordering time.
b	Backlogging parameter.
t_d	The length of time in which the product remains fresh because of no deterioration.
t_s	The length of the on-hand inventory period.
θ	Deterioration rate of the product.
M	Credit period offered by the supplier to retailer.
Q	Retailer's Order quantity per cycle.
Q_r	Retailer's initial ordering quantity.
α	Effectiveness parameter for the credit period.
η	Maximum rate of demand due to the advertisement.
β	Demand sensitivity of customer about c_r.

Decision variables

Q_0	Retailer's normal ordering quantity when there is no credit period.
A_d	Number of advertisements.
T	Length of the inventory cycle.

3.2. Assumptions

The following assumptions are made throughout the paper to formulate the model.

(i) The Covid-19 pandemic is affecting the whole world. This has disrupted the normal life of people. One of the ways to prevent this epidemic is a lockdown. As a result of this lockdown, people have lost the spontaneity of living a normal life and the demand for things that people need to go out has decreased a bit. On the other hand, the use of things like masks, sanitizer etc., have increased. Also, in this situation, many times, people buy a lot of their daily necessities for fear of falling into lockdown. In addition, the lockdown has resulted in the closure of various industries and factories which has hampered people's income. So, people are often unable to buy things according to their demand due to financial crisis. So, in this pandemic situation, we can not accurately determine the demand for any item in the market. For all these reasons, we have assumed that the demand rate of the inventory system is random.

(ii) Advertising is one of the ways to accept a product to the public and henceforth increase its demand. This advertisement makes the product attractive to the customer, thereby increasing the interest of the customer to buy the product. Also, continuous advertisement makes the customer monotonous, it may have a bad impact on demand. The demand of today's conscious customers depends not only on advertisements but also on retail prices. No matter how attractive the advertisement is, if the retail price is high, it will have a bad effect on demand. In this respect, we consider a demand function in the following way: $D(A_d, c_r) = D_0\left[1 + \frac{(\eta-1)A_d}{A_d^{\max}}\right]^k - \beta c_r$, where $0 \leq k \leq 1$ and D_0 be the normal market demand, which is assumed to be stochastic.

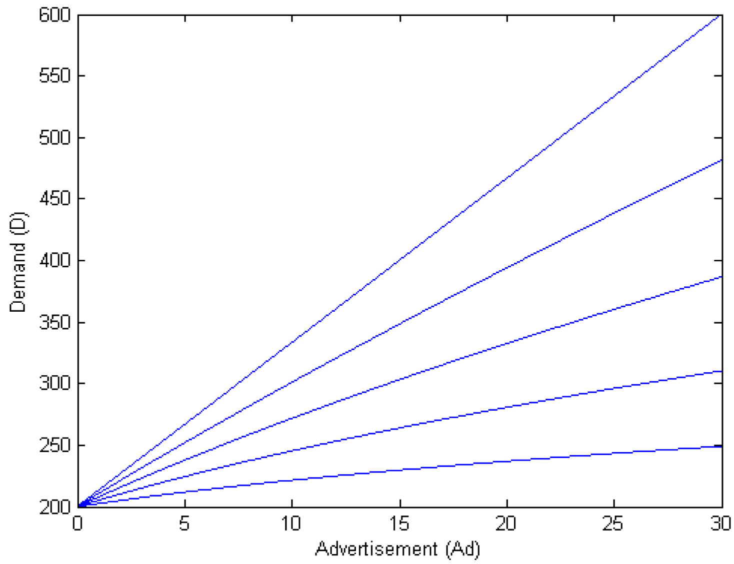


FIGURE 2. A_d *vs.* D.

(iii) It is noted that when a retailer retails his/her products to maintain its profit, he/she charges a price higher than its procurement price. So, we express the retailer's retail price as $c_r = mc_p$, where m is the markup. Though the retailer will want to cover some of the cost of the advertisement through this markup but, if the markup is too high, the retail price increase will have a bad impact on the customers demand. So, keeping all this in mind, we consider the markup as: $m = m_1 + \gamma A_d$, where $m_1 > 0$ be the normal markup when there is no advertisement and γ is the maximum markup when the retailer is able to maximum advertise.

(iv) Due to undetermined demand, the retailer imports approximate quantities of goods from suppliers. So, there is often a shortage of goods. As a result, retailers fail to meet the demand of some customers. This unsatisfied demand is assumed to be partially backlogged. The amount of backorders increases with the decreasing customer's waiting time $(T - t)$, which depends somewhat on the retailer's good behaviour. The rate of partial backorder is, $e^{-b(T-t)}$, where $b > 0$ is the backlogging parameter.

(v) Here, we discuss about non instantaneous deteriorating items, where products will deteriorate after the time t_d. So, in the interval $[0,\ t_d\ ]$, the product remains fresh and the deterioration occurs at a constant rate θ during the time interval $[t_d,\ T]$.

(vi) To help the retailer during the financial crisis and also attract the retailer's point of view, a supplier gives a credit period M to the retailer. The length of this credit period depends on the retailer's ordering quantity. So, due to getting this credit period, the retailer wants to increase his/her ordering quantity depending on credit period M. Hence, the retailer's initial ordering quantity is, $Q_r = Q_0 e^{\alpha M}$, where Q_0 is the retailer's normal ordering quantity when there is no offering of credit period and $\alpha(\geq 0)$ is the effectiveness parameter for the credit period.

(vii) In this credit policy, the retailer gives δ portion of his/her total purchase cost at the purchasing time and the remaining $(1 - \delta)$ portion will be paid at the end of the credit period M.

(viii) Though the supplier gives a credit period M to the retailer, he/she will pay his/her purchase cost after all inventory is sold because if all products of the retailer are not sold within the credit period, he/she has to take a loan from the bank to pay the supplier at the end of the credit period.

(ix) The time horizon is infinite.

4. Model formulation

4.1. Mathematical formulation for supplier

Supplier's total profits contains purchasing costs, sales revenue, opportunity loss.

Since Q is the total quantity and p is the purchase cost per unit item, then the *supplier's total purchase cost (SPC)* is given by:

$$\mathrm{SPC} = \int_0^{\infty} pQf(D_0)\mathrm{d}D_0$$

The *supplier's sales revenue (SSR)* is given by:

$$\mathrm{SSR} = \int_0^{\infty} c_pQf(D_0)\mathrm{d}D_0$$

Since, the supplier offers a credit period M to the retailer, but during this period the supplier faces an interest loss at a rate I_l. So the *supplier's opportunity loss (SOL)* is obtained as:

$$\mathrm{SOL} = \int_0^{\infty} [I_l(1-\delta)c_pQM]f(D_0)\mathrm{d}D_0$$

Hence, the *total expected net profit of supplier (TPS (Q))* is given by:

$$\mathrm{TPS}(Q) = \frac{1}{T}\int_0^{\infty} [c_pQ - pQ - I_l(1-\delta)c_pQM]f(D_0)\mathrm{d}D_0 \tag{1}$$

4.2. Mathematical formulation for retailer

In this model, the retailer purchases the non-instantaneous deteriorating item from the supplier, whose deterioration starts after the time t_d. In this case, the products remains fresh during the time interval $[0, t_d]$. So, the inventory level during the period $[0, t_d]$ decreases only with the demand rate D. After this time period, deterioration started at a constant rate θ and the inventory level drops to zero at $t = t_s$ due to both demand rate and the effect of deterioration. Then, the entire demand is partially backlogged in the period $[t_s, T]$.

Based on the preceding assumption this model is formulated as follows:

The inventory at time intervals $[0, T]$ can be represented by the following differential equations:

$$\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = -D, \text{ with } I_1(0) = Q_0e^{\alpha M}, \quad 0 \le t \le t_d \tag{2}$$

$$\frac{\mathrm{d}I_2(t)}{\mathrm{d}t} + \theta I_2(t) = -D, \text{ with } I_2(t_s) = 0, \qquad t_d \le t \le t_s \tag{3}$$

$$\frac{\mathrm{d}I_3(t)}{\mathrm{d}t} = -De^{-b(T-t)}, \text{ with } I_3(t_s) = 0 \quad t_s \le t \le T \tag{4}$$

Since $I_1(t) = Q_0e^{\alpha M}$ at $t = 0$ and using this boundary condition, we have the solution of equation (2) as:

$$I_1(t) = -Dt + Q_0e^{\alpha M} \qquad 0 \le t \le t_d \tag{5}$$

Since $I_2(t) = 0$ at $t = t_s$ and using this condition, we have the solution of equation (3) as:

$$I_2(t) = \frac{D}{\theta}\left[e^{\theta(t_s-t)} - 1\right] \qquad t_d \le t \le t_s \tag{6}$$

Again since $I_3(t) = 0$ at $t = t_s$, using this boundary condition, we have the solution of equation (4) is:

$$I_3(t) = \frac{D}{b}\left[e^{-b(T-t_s)} - e^{-b(T-t)}\right] \qquad t_s \le t \le T \tag{7}$$

Therefore, the *maximum shortage* (S) is given by:

$$\begin{aligned} S &= -I_3(T) \\ &= \frac{D}{b}\left[1 - e^{-b(T-t_s)}\right] \end{aligned} \tag{8}$$

So, finally the total initial stock and the maximum shortage make the *total order quantity* (Q) for each period obtained as:

$$\begin{aligned} Q &= I_1(0) + S \\ &= Q_0 e^{\alpha M} + \frac{D}{b}\left[1 - e^{-b(T-t_s)}\right] \end{aligned} \tag{9}$$

Lemma 1. $Q_0 e^{\alpha M} = Dt_d + \frac{D}{\theta}\left[e^{\theta(t_s-t_d)} - 1\right]$

Proof. Since, t_d is the ending point of inventory level $I_1(t)$ and the starting point of $I_2(t)$, at this point $I_1(t)$ and $I_2(t)$ will be equal. So, we can write

$$\begin{aligned} I_1(t_d) &= I_2(t_d) \\ \text{or, } -Dt_d + Q_0 e^{\alpha M} &= \frac{D}{\theta}\left[e^{\theta(t_s-t_d)} - 1\right] \\ \text{or, } Q_0 e^{\alpha M} &= Dt_d + \frac{D}{\theta}\left[e^{\theta(t_s-t_d)} - 1\right] \end{aligned} \tag{10}$$

Hence the proof. □

Purchase cost:

Since Q is the total order quantity and c_p is the purchase cost per unit item, then the *total purchase cost (PC)* is given by:

$$\begin{aligned} \mathrm{PC} &= \int_0^\infty (c_p Q) f(D_0)\,\mathrm{d}D_0 \\ &= c_p \int_0^\infty \left[Q_0 e^{\alpha M} + \frac{D}{b}\left[1 - e^{-b(T-t_s)}\right]\right] f(D_0)\,\mathrm{d}D_0 \end{aligned}$$

Holding cost:

After purchasing the products from the supplier, the retailer does not sell all products at once. So, he/she has to maintain this remaining inventory for a certain period. This maintaining cost are known as holding cost. So, the retailer's *holding cost (HC)* is given by:

$$\begin{aligned} \mathrm{HC} &= c_h \int_0^\infty \left[\int_0^{t_d} I_1(t)\,\mathrm{d}t + \int_{t_d}^{t_s} I_2(t)\,\mathrm{d}t\right] f(D_0)\,\mathrm{d}D_0 \\ &= c_h \int_0^\infty \left[\int_0^{t_d} (-Dt + Q_0 e^{\alpha M})\,\mathrm{d}t + \int_{t_d}^{t_s} \frac{D}{\theta}[e^{\theta(t_s-t)} - 1]\,\mathrm{d}t\right] f(D_0)\,\mathrm{d}D_0 \\ &= c_h \int_0^\infty \left[-\frac{Dt_d^2}{2} + t_d Q_0 e^{\alpha M} + \frac{D}{\theta}(t_d - t_s) + \frac{D}{\theta^2}\left\{e^{\theta(t_s-t_d)} - 1\right\}\right] f(D_0)\,\mathrm{d}D_0 \end{aligned}$$

As a result of the shortage, some customers will wait for the next lot to buy the product from the retailer and some customers will go to an alternative store to fulfil their demand. This means that there is some unsatisfied demand and some opportunity loss arises. The value of this unsatisfied demand is known as backorder cost and value for loss are called Lost sales.

Backorder cost:

The mathematical form of *backorder cost (BC)* is given as:

$$\begin{aligned}
\mathrm{BC} &= c_b \int_0^\infty \left[\int_{t_s}^T -I_3(t)\,\mathrm{d}t \right] f(D_0)\,\mathrm{d}D_0 \\
&= -c_b \int_0^\infty \left[\int_{t_s}^T \frac{D}{b}[e^{-b(T-t_s)} - e^{-b(T-t)}]\,\mathrm{d}t \right] f(D_0)\,\mathrm{d}D_0 \\
&= \frac{c_b D}{b^2} \int_0^\infty \left[1 - e^{-b(T-t_s)}\{1 + b(T-t_s)\} \right] f(D_0)\,\mathrm{d}D_0
\end{aligned}$$

Lost sale:

The mathematical form of *lost sale (LC)* during the time interval $[t_s, T]$ is given by:

$$\begin{aligned}
\mathrm{LC} &= c_l \int_0^\infty \left[\int_{t_s}^T D\left\{1 - e^{-b(T-t)}\right\}\mathrm{d}t \right] f(D_0)\,\mathrm{d}D_0 \\
&= \frac{c_l D}{b} \int_0^\infty \left[b(T-t_s) + e^{-b(T-t_s)} - 1 \right] f(D_0)\,\mathrm{d}D_0
\end{aligned}$$

Sales revenue:

Since the selling price of the retailer is c_r and the positive stock period is t_s with random demand D and the maximum shortage S, so the total retailer's *sales revenue (RSR)* for the whole cycle length is given by:

$$\begin{aligned}
\mathrm{RSR} &= \int_0^\infty [Dc_r t_s + c_r S] f(D_0)\,\mathrm{d}D_0 \\
&= \int_0^\infty Dc_r \left[t_s + \frac{1}{b}\left\{1 - e^{-b(T-t_s)}\right\} \right] f(D_0)\,\mathrm{d}D_0
\end{aligned}$$

Advertising cost:

Due to giving advertisements, the retailer has to bear some of the advertising costs. He/she gives A_d number of advertisements and c_a is the advertising cost per advertising, then the total *advertising cost (AC)* is given as:

$$\mathrm{AC} = c_a A_d$$

Here we discuss different cases for different length of M.

4.2.1. Case 1: When $0 < M \leq t_d$

Due to getting partial credit period, the retailer will be able to earn interest on $(1-\delta)$ portion. This interest earn depends on customers demand and interest rate I_e.

The *interest earned* (IE_{11}) is given by:

$$\begin{aligned}
\mathrm{IE}_{11} &= \int_0^\infty \left[(1-\delta)Dc_r I_e \int_0^{t_s} (T-t)\,\mathrm{d}t \right] f(D_0)\,\mathrm{d}D_0 \\
&= \int_0^\infty \left[(1-\delta)Dc_r I_e t_s \left(T - \frac{t_s}{2} \right) \right] f(D_0)\,\mathrm{d}D_0
\end{aligned}$$

If the retailer can not pay the supplier within the credit period M, then after M, the retailer is charged an interest by the supplier. This interest charged depends on the available stock during this interval and interest rate I_c.

The *interest charged* (IC_{11}) is given by:

$$\begin{aligned}
\mathrm{IC}_{11} &= \int_0^\infty c_p I_c \Big[\int_M^{t_d} I_1(t)\,\mathrm{d}t + \int_{t_d}^{t_s} I_2(t)\,\mathrm{d}t\Big] f(D_0)\,\mathrm{d}D_0 \\
&= \int_0^\infty c_p I_c \Big[\int_M^{t_d} \left(-Dt + Q_0 e^{\alpha M}\right)\mathrm{d}t + \int_{t_d}^{t_s} \frac{D}{\theta}\left[e^{\theta(t_s-t)} - 1\right]\mathrm{d}t\Big] f(D_0)\,\mathrm{d}D_0 \\
&= \int_0^\infty \Big[c_p I_c \Big\{\frac{D}{2}\left(M^2 - t_d^2\right) + Q_0 e^{\alpha M}(t_d - M) - \frac{D}{\theta^2}(1 - e^{\theta(t_s-t_d)}) - \frac{D}{\theta}(t_s - t_d)\Big\}\Big] f(D_0)\,\mathrm{d}D_0
\end{aligned}$$

The *retailer's expected net profit* ($\mathrm{TPR}_{11}(Q_0, A_d, T)$) is given by:

$$\begin{aligned}
\mathrm{TPR}_{11}(Q_0, A_d, T) &= \frac{1}{T}[\mathrm{RSR} - \mathrm{OC} - \mathrm{PC} - \mathrm{HC} - \mathrm{BC} - \mathrm{LC} - \mathrm{IP} - \mathrm{AC} + \mathrm{IE}_{21} - \mathrm{IC}_{21}] \\
&= \frac{1}{T}\int_0^\infty \Big[Dc_r\Big\{t_s + \frac{1}{b}\Big[1 - e^{-b(T-t_s)}\Big]\Big\} - A - c_p\Big[Q_0 e^{\alpha M} + \frac{D}{b}\Big[1 - e^{-b(T-t_s)}\Big]\Big] \\
&\quad - c_h\Big[-\frac{Dt_d^2}{2} + t_d Q_0 e^{\alpha M} + \frac{D}{\theta}(t_d - t_s) + \frac{D}{\theta^2}\Big\{e^{\theta(t_s-t_d)} - 1\Big\}\Big] \\
&\quad - \frac{c_b D}{b^2}\Big[1 - e^{-b(T-t_s)}\{1 + b(T - t_s)\}\Big] - \frac{c_l D}{b}\Big[b(T - t_s) + e^{-b(T-t_s)} - 1\Big] \\
&\quad - i_c \delta c_p Q t_s - c_a A_d + \Big[(1-\delta)Dc_r I_e t_s\Big(T - \frac{t_s}{2}\Big)\Big] - \Big[c_p I_c\Big\{\frac{D}{2}\left(M^2 - t_d^2\right) \\
&\quad + Q_0 e^{\alpha M}(t_d - M) - \frac{D}{\theta^2}\Big(1 - e^{\theta(t_s-t_d)}\Big) - \frac{D}{\theta}(t_s - t_d)\Big\}\Big]\Big] f(D_0)\,\mathrm{d}D_0
\end{aligned} \tag{11}$$

The *expected integrated profit* ($\mathrm{TPI}_{11}(Q_0, A_d, T)$) is given by:

$$\begin{aligned}
\mathrm{TPI}_{11}(Q_0, A_d, T) &= \mathrm{TPS} + \mathrm{TPR}_{11} \\
&= \frac{1}{T}\int_0^\infty \Big[(c_p Q - pQ - I_l(1-\delta)c_p QM) + Dc_r\Big\{t_s + \frac{1}{b}\Big[1 - e^{-b(T-t_s)}\Big]\Big\} - A \\
&\quad - c_p\Big[Q_0 e^{\alpha M} + \frac{D}{b}\Big[1 - e^{-b(T-t_s)}\Big]\Big] - c_h\Big[-\frac{Dt_d^2}{2} + t_d Q_0 e^{\alpha M} + \frac{D}{\theta}(t_d - t_s) \\
&\quad + \frac{D}{\theta^2}\Big\{e^{\theta(t_s-t_d)} - 1\Big\}\Big] - \frac{c_b D}{b^2}\Big[1 - e^{-b(T-t_s)}\{1 + b(T - t_s)\}\Big] \\
&\quad - \frac{c_l D}{b}\Big[b(T - t_s) + e^{-b(T-t_s)} - 1\Big] - i_c \delta c_p Q t_s - c_a A_d + \Big[(1-\delta)Dc_r I_e t_s\Big(T - \frac{t_s}{2}\Big)\Big] \\
&\quad - \Big[c_p I_c\Big\{\frac{D}{2}\left(M^2 - t_d^2\right) + Q_0 e^{\alpha M}(t_d - M) - \frac{D}{\theta^2}\Big(1 - e^{\theta(t_s-t_d)}\Big) \\
&\quad - \frac{D}{\theta}(t_s - t_d)\Big\}\Big]\Big] f(D_0)\,\mathrm{d}D_0
\end{aligned} \tag{12}$$

4.2.2. *Case 2: When* $t_d < M \le t_s$

The *interest earned* (IE_{12}) is given by:

$$\begin{aligned}
\mathrm{IE}_{12} &= \mathrm{IE}_{11} \\
&= \int_0^\infty \Big[(1-\delta)Dc_r I_e t_s\Big(T - \frac{t_s}{2}\Big)\Big] f(D_0)\,\mathrm{d}D_0
\end{aligned}$$

The *interest charged* (IC_{12}) is given by:

$$\begin{aligned}
\mathrm{IC}_{12} &= \int_0^\infty \left[c_p I_c \int_M^{t_s} I_2(t)\,\mathrm{d}t\right] f(D_0)\,\mathrm{d}D_0 \\
&= \int_0^\infty \left[c_p I_c \int_M^{t_s} \frac{D}{\theta}\left[e^{\theta(t_s-t)} - 1\right]\mathrm{d}t\right] f(D_0)\mathrm{d}D_0 \\
&= \int_0^\infty \left[c_p I_c \left\{\frac{D}{\theta^2}\left(e^{\theta(t_s-M)} - 1\right) + \frac{D}{\theta}(M - t_s)\right\}\right] f(D_0)\,\mathrm{d}D_0
\end{aligned}$$

The *retailer's expected net profit* ($TPR_{12}(Q_0, A_d, T)$) is given by:

$$\begin{aligned}
\mathrm{TPR}_{12}(Q_0, A_d, T) &= \frac{1}{T}[\mathrm{RSR} - \mathrm{OC} - \mathrm{PC} - \mathrm{HC} - \mathrm{BC} - \mathrm{LC} - \mathrm{IP} - \mathrm{AC} + \mathrm{IE}_{22} - \mathrm{IC}_{22}] \\
&= \frac{1}{T}\int_0^\infty \left[Dc_r\left\{t_s + \frac{1}{b}\left[1 - e^{-b(T-t_s)}\right]\right\} - A - c_p\left[Q_0 e^{\alpha M} + \frac{D}{b}\left[1 - e^{-b(T-t_s)}\right]\right]\right. \\
&\quad - c_h\left[-\frac{Dt_d^2}{2} + t_d Q_0 e^{\alpha M} + \frac{D}{\theta}(t_d - t_s) + \frac{D}{\theta^2}\left\{e^{\theta(t_s-t_d)} - 1\right\}\right] \\
&\quad - \frac{c_b D}{b^2}\left[1 - e^{-b(T-t_s)}\{1 + b(T - t_s)\}\right] - \frac{c_l D}{b}\left[b(T - t_s) + e^{-b(T-t_s)} - 1\right] \\
&\quad - i_c \delta c_p Q t_s - c_a A_d + \left[(1-\delta)Dc_r I_e t_s\left(T - \frac{t_s}{2}\right)\right] - \left[c_p I_c\left\{\frac{D}{\theta^2}\left(e^{\theta(t_s-M)} - 1\right)\right.\right. \\
&\quad \left.\left.\left. + \frac{D}{\theta}(M - t_s)\right\}\right]\right] f(D_0)\,\mathrm{d}D_0
\end{aligned} \tag{13}$$

The *expected integrated profit* ($\mathrm{TPI}_{12}(Q_0, A_d, T)$) is given by:

$$\begin{aligned}
\mathrm{TPI}_{12}(Q_0, A_d, T) &= \mathrm{TPS} + \mathrm{TPR}_{12} \\
&= \frac{1}{T}\int_0^\infty \left[(c_p Q - pQ - I_l(1-\delta)c_p QM) + Dc_r\left\{t_s + \frac{1}{b}\left[1 - e^{-b(T-t_s)}\right]\right\} - A\right. \\
&\quad - c_p\left[Q_0 e^{\alpha M} + \frac{D}{b}\left[1 - e^{-b(T-t_s)}\right]\right] - c_h\left[-\frac{Dt_d^2}{2} + t_d Q_0 e^{\alpha M} + \frac{D}{\theta}(t_d - t_s)\right. \\
&\quad \left. + \frac{D}{\theta^2}\left\{e^{\theta(t_s-t_d)} - 1\right\}\right] - \frac{c_b D}{b^2}\left[1 - e^{-b(T-t_s)}\{1 + b(T - t_s)\}\right] - \frac{c_l D}{b}\left[b(T - t_s)\right. \\
&\quad \left. + e^{-b(T-t_s)} - 1\right] - i_c \delta c_p Q t_s - c_a A_d + \left[(1-\delta)Dc_r I_e t_s\left(T - \frac{t_s}{2}\right)\right] \\
&\quad \left. - \left[c_p I_c\left\{\frac{D}{\theta^2}\left(e^{\theta(t_s-M)} - 1\right) + \frac{D}{\theta}(M - t_s)\right\}\right]\right] f(D_0)\,\mathrm{d}D_0
\end{aligned} \tag{14}$$

4.2.3. Case 3: When $t_s < M \leq T$

The *interest earned* (IE_{13}) is given by:

$$\begin{aligned}
\mathrm{IE}_{13} &= \mathrm{IE}_{11} \\
&= \int_0^\infty \left[(1-\delta)Dc_r I_e t_s\left(T - \frac{t_s}{2}\right)\right] f(D_0)\,\mathrm{d}D_0
\end{aligned}$$

Since there is no positive stock after M. Therefore the *interest charged* (IC_{13}) is given by:

$$\mathrm{IC}_{13} = 0$$

The *retailer's expected net profit* ($\mathrm{TPR}_{13}(Q_0, A_d, T)$) is given by:

$$
\begin{aligned}
\mathrm{TPR}_{13}(Q_0, A_d, T) &= \frac{1}{T}[\mathrm{RSR} - \mathrm{OC} - \mathrm{PC} - \mathrm{HC} - \mathrm{BC} - \mathrm{LC} - \mathrm{IP} - \mathrm{AC} + \mathrm{IE}_{23} - \mathrm{IC}_{23}] \\
&= \frac{1}{T}\int_0^{\infty}\Bigg[Dc_r\left\{t_s + \frac{1}{b}\left[1 - e^{-b(T-t_s)}\right]\right\} - A - c_p\left[Q_0 e^{\alpha M} + \frac{D}{b}\left[1 - e^{-b(T-t_s)}\right]\right] \\
&\quad - c_h\left[-\frac{Dt_d^2}{2} + t_d Q_0 e^{\alpha M} + \frac{D}{\theta}(t_d - t_s) + \frac{D}{\theta^2}\left\{e^{\theta(t_s - t_d)} - 1\right\}\right] \\
&\quad - \frac{c_b D}{b^2}\left[1 - e^{-b(T-t_s)}\{1 + b(T - t_s)\}\right] - \frac{c_l D}{b}\left[b(T - t_s) + e^{-b(T-t_s)} - 1\right] \\
&\quad - i_c\delta c_p Q t_s - c_a A_d + \left[(1-\delta)Dc_r I_e t_s\left(T - \frac{t_s}{2}\right)\right]\Bigg] f(D_0)\,\mathrm{d}D_0
\end{aligned} \tag{15}
$$

The *expected integrated profit* ($\mathrm{TPI}_{13}(Q_0, A_d, T)$) is given by:

$$
\begin{aligned}
\mathrm{TPI}_{13}(Q_0, A_d, T) &= \mathrm{TPS} + \mathrm{TPR}_{13} \\
&= \frac{1}{T}\int_0^{\infty}\Bigg[(c_p Q - pQ - I_l(1-\delta)c_p Q M) + Dc_r\left\{t_s + \frac{1}{b}\left[1 - e^{-b(T-t_s)}\right]\right\} - A \\
&\quad - c_p\left[Q_0 e^{\alpha M} + \frac{D}{b}\left[1 - e^{-b(T-t_s)}\right]\right] - c_h\Bigg[-\frac{Dt_d^2}{2} + t_d Q_0 e^{\alpha M} + \frac{D}{\theta}(t_d - t_s) \\
&\quad + \frac{D}{\theta^2}\left\{e^{\theta(t_s - t_d)} - 1\right\}\Bigg] - \frac{c_b D}{b^2}\left[1 - e^{-b(T-t_s)}\{1 + b(T - t_s)\}\right] \\
&\quad - \frac{c_l D}{b}\left[b(T - t_s) + e^{-b(T-t_s)} - 1\right] - i_c\delta c_p Q t_s - c_a A_d \\
&\quad + \left[(1-\delta)Dc_r I_e t_s\left(T - \frac{t_s}{2}\right)\right]\Bigg] f(D_0)\,\mathrm{d}D_0
\end{aligned} \tag{16}
$$

For numerical illustration, we are considering a particular density function for the normal market demand D_0. Here, we consider a uniform distribution for interval $[a, c]$, where a is the minimum amount of demand and c is the maximum amount of demand. So the *continuous uniform distribution demand rate* can be defined as:

$$
f(D_0) = \begin{cases} \dfrac{1}{c-a}; & \text{when} \quad a \le D_0 \le c \\ 0; & \text{otherwise} \end{cases}
$$

Under the demand rate, the expected integrated profit ($TPI(Q_0, A_d, T)$) is given as follows:

<u>Case 1</u>: When $0 < M \le t_d$

$$
\begin{aligned}
\mathrm{TPI}_{11}(Q_0, A_d, T) &= \frac{1}{T}\Bigg[\left\{\frac{a+c}{2}\left[1 + \frac{(\eta-1)A_d}{A_d^{\max}}\right]^k - \beta c_r\right\}\Bigg\{c_r\left[t_s + \frac{1}{b}\left\{1 - e^{-b(T-t_s)}\right\}\right] \\
&\quad + c_h\left[\frac{t_d^2}{2} - \frac{1}{\theta}(t_d - t_s) - \frac{1}{\theta^2}\left\{e^{\theta(t_s - t_d)} - 1\right\}\right] - \frac{c_b}{b^2}\left[1 - e^{-b(T-t_s)}\{1 + b(T - ts)\}\right] \\
&\quad - \frac{c_l}{b}\left[b(T - t_s) + e^{-b(T-t_s)} - 1\right] - i_c\delta c_p t_s\frac{1}{b}\left\{1 - e^{-b(T-t_s)}\right\} + (1-\delta)c_r I_e t_s\left(T - \frac{t_s}{2}\right) \\
&\quad - c_p I_c\left[\frac{1}{2}(M^2 - t_d^2) - \frac{1}{\theta^2}\left\{1 - e^{\theta(t_s - t_d)}\right\} - \frac{1}{\theta}(t_s - t_d)\right] - \frac{1}{b}[p + I_l(1-\delta)c_p M] \\
&\quad \left\{1 - e^{-b(T-t_s)}\right\}\Bigg\} - Q_0 e^{\alpha M}\{p + c_h t_d + c_p I_c(t_d - M) + I_l(1-\delta)c_p M - i_c\delta c_p t_s\} \\
&\quad - A - c_a A_d\Bigg]
\end{aligned} \tag{17}
$$

Case 2: When $t_d < M \le t_s$

$$\begin{aligned}\mathrm{TPI}_{12}(Q_0,A_d,T)=\frac{1}{T}\Bigg[&\Bigg\{\frac{a+c}{2}\left[1+\frac{(\eta-1)A_d}{A_d^{\max}}\right]^k-\beta c_r\Bigg\}\Bigg\{c_r\left[t_s+\frac{1}{b}\Big\{1-e^{-b(T-t_s)}\Big\}\right]\\&+c_h\left[\frac{t_d^2}{2}-\frac{1}{\theta}(t_d-t_s)-\frac{1}{\theta^2}\Big\{e^{\theta(t_s-t_d)}-1\Big\}\right]-\frac{c_b}{b^2}\left[1-e^{-b(T-t_s)}\{1+b(T-ts)\}\right]\\&-\frac{c_l}{b}\left[b(T-t_s)+e^{-b(T-t_s)}-1\right]-i_c\delta c_p t_s\frac{1}{b}\Big\{1-e^{-b(T-t_s)}\Big\}+(1-\delta)c_rI_et_s\left(T-\frac{t_s}{2}\right)\\&-c_pI_c\left[\frac{1}{\theta^2}\left(e^{\theta(t_s-M)}-1\right)+\frac{1}{\theta}(M-t_s)\right]-\frac{1}{b}[p+I_l(1-\delta)c_pM]\Big\{1-e^{-b(T-t_s)}\Big\}\Bigg\}\\&-Q_0e^{\alpha M}\{p+c_ht_d+I_l(1-\delta)c_pM-i_c\delta c_pt_s\}-A-c_aA_d\Bigg]\end{aligned}\tag{18}$$

Case 3: When $t_s < M \le T$

$$\begin{aligned}\mathrm{TPI}_{13}(Q_0,A_d,T)=\frac{1}{T}\Bigg[&\Bigg\{\frac{a+c}{2}\left[1+\frac{(\eta-1)A_d}{A_d^{\max}}\right]^k-\beta c_r\Bigg\}\Bigg\{c_r\left[t_s+\frac{1}{b}\Big\{1-e^{-b(T-t_s)}\Big\}\right]\\&+c_h\left[\frac{t_d^2}{2}-\frac{1}{\theta}(t_d-t_s)-\frac{1}{\theta^2}\Big\{e^{\theta(t_s-t_d)}-1\Big\}\right]-\frac{c_b}{b^2}\left[1-e^{-b(T-t_s)}\{1+b(T-ts)\}\right]\\&-\frac{c_l}{b}\left[b(T-t_s)+e^{-b(T-t_s)}-1\right]-i_c\delta c_p t_s\frac{1}{b}\Big\{1-e^{-b(T-t_s)}\Big\}+(1-\delta)c_rI_et_s\left(T-\frac{t_s}{2}\right)\\&-\frac{1}{b}[p+I_l(1-\delta)c_pM]\Big\{1-e^{-b(T-t_s)}\Big\}\Bigg\}-Q_0e^{\alpha M}\{p+c_ht_d+I_l(1-\delta)c_pM\\&-i_c\delta c_pt_s\}-A-c_aA_d\Bigg]\end{aligned}\tag{19}$$

Equation (17) can be written as:

$$\begin{aligned}\mathrm{TPI}_{11}(Q_0,A_d,T)=\frac{1}{T}\Bigg[&D\Bigg\{\left[1-e^{-b(T-t_s)}\right]X-c_l(T-t_s)+\frac{c_b}{b}(T-t_s)e^{-b(T-t_s)}+(1-\delta)c_rI_et_s\left(T-\frac{t_s}{2}\right)\\&+Y-\frac{1}{b}\Big\{1-e^{-b(T-t_s)}\Big\}i_c\delta c_pt_s+\frac{1}{\theta^2}(c_h+c_pI_c)\Big\{\theta(t_s-t_d)-e^{\theta(t_s-t_d)}\Big\}+c_rt_s\Bigg\}\\&+Q_0e^{\alpha M}(i_c\delta c_pt_s-Z)-A-c_aA_d\Bigg]\end{aligned}\tag{20}$$

Where, $X=\frac{1}{b}[c_r+c_l-p-I_l(1-\delta)c_pM]-\frac{c_b}{b^2}$
$Y=\frac{1}{2}c_ht_d^2+\frac{1}{\theta^2}(c_h+c_pI_c)-\frac{1}{2}c_pI_c(M^2-t_d^2)$
$Z=p+c_ht_d+c_pI_c(t_d-M)+I_l(1-\delta)c_pM$

5. Solution procedure

In our proposed model, there are three decision variable Q_0, A_d and T, where A_d is an integer variable and Q_0, T are real variables. So, analytic solutions of this model are not practicable. For this purpose, to get optimal solution of our problem we first consider A_d as constant and then differentiate the equation (20) with respect to Q_0 and T respectively.

The first order and second order derivatives of TPI_{11} with respect to Q_0 and T when A_d considering as a constant are as follows:

$$\begin{aligned}
\frac{\partial \mathrm{TPI}_{11}}{\partial Q_0} &= \frac{1}{T}\Bigg[D\Bigg\{-Xbe^{-b(T-t_s)}\frac{\partial t_s}{\partial Q_0} + c_l\frac{\partial t_s}{\partial Q_0} - \frac{c_b}{b}e^{-b(T-t_s)}\frac{\partial t_s}{\partial Q_0} + c_b(T-t_s)e^{-b(T-t_s)}\frac{\partial t_s}{\partial Q_0}\\
&\quad + (1-\delta)c_rI_e\left(T-\frac{t_s}{2}\right)\frac{\partial t_s}{\partial Q_0} - \frac{1}{2}(1-\delta)c_rI_et_s\frac{\partial t_s}{\partial Q_0} - \frac{1}{b}\left\{1-e^{-b(T-t_s)}\right\}i_c\delta c_p\frac{\partial t_s}{\partial Q_0}\\
&\quad + i_c\delta c_pt_se^{-b(T-t_s)}\frac{\partial t_s}{\partial Q_0} + \frac{1}{\theta}(c_h+c_pI_c)\left\{\frac{\partial t_s}{\partial Q_0} - e^{\theta(t_s-t_d)}\frac{\partial t_s}{\partial Q_0}\right\} + c_r\frac{\partial t_s}{\partial Q_0}\Bigg\}\\
&\quad + e^{\alpha M}(i_c\delta c_pt_s - Z) + Q_0e^{\alpha M}i_c\delta c_p\frac{\partial t_s}{\partial Q_0}\Bigg]
\end{aligned} \tag{21}$$

$$\begin{aligned}
\frac{\partial^2 \mathrm{TPI}_{11}}{\partial Q_0^2} &= \frac{1}{T}\Bigg[Dbe^{-b(T-t_s)}\left(\frac{\partial t_s}{\partial Q_0}\right)^2\left\{-Xb - \frac{c_b}{b} + c_b(T-t_s) + \frac{1}{b}i_c\delta c_p(1+bt_s)\right\}\\
&\quad + De^{-b(T-t_s)}[i_c\delta c_p - c_b]\left(\frac{\partial t_s}{\partial Q_0}\right)^2 + De^{-b(T-t_s)}\Bigg\{-Xb - \frac{c_b}{b} + c_b(T-t_s) + \frac{1}{b}i_c\delta\, c_p\\
&\quad (1+bt_s)\Bigg\}\frac{\partial^2 t_s}{\partial Q_0^2} - D(1-\delta)I_ec_r\left(\frac{\partial t_s}{\partial Q_0}\right)^2 - (c_h+c_pI_c)e^{\theta(t_s-t_d)}\left(\frac{\partial t_s}{\partial Q_0}\right)^2 + 2e^{\alpha M}i_c\delta c_p\frac{\partial t_s}{\partial Q_0}\\
&\quad + \Bigg\{D\Bigg[c_l + (1-\delta)c_rI_e\left(T-\frac{t_s}{2}\right) - \frac{1}{2}(1-\delta)c_rI_et_s - \frac{1}{b}i_c\delta c_p + \frac{1}{\theta}(c_h+c_pI_c)\left(1-e^{\theta(t_s-t_d)}\right)\\
&\quad + c_r\Bigg] + Q_0e^{\alpha M}i_c\delta c_p\Bigg\}\frac{\partial^2 t_s}{\partial Q_0^2}\Bigg]
\end{aligned} \tag{22}$$

$$\begin{aligned}
\frac{\partial^2 \mathrm{TPI}_{11}}{\partial T\partial Q_0} &= -\frac{1}{T^2}\Bigg[De^{-b(T-t_s)}\left\{-Xb - \frac{c_b}{b} + c_b(T-t_s) + \frac{1}{b}i_c\delta c_p(1+bt_s)\right\}\frac{\partial t_s}{\partial Q_0}\\
&\quad + \Bigg\{D\Bigg[c_l + (1-\delta)c_rI_e(T-t_s) - \frac{1}{b}i_c\delta c_p + \frac{1}{\theta}(c_h+I_cc_p)\left(1-e^{\theta(t_s-t_d)}\right) + c_r\Bigg]\\
&\quad + Q_0e^{\alpha M}i_c\delta c_p\Bigg\}\frac{\partial t_s}{\partial Q_0} + e^{\alpha M}(i_c\delta c_pt_s - Z)\Bigg] + \frac{1}{T}\Bigg[-Dbe^{-b(T-t_s)}\Bigg\{-Xb - \frac{c_b}{b} + c_b(T-t_s)\\
&\quad + \frac{1}{b}i_c\delta c_p(1+bt_s)\Bigg\}\frac{\partial t_s}{\partial Q_0} + Dc_be^{-b(T-t_s)} + D(1-\delta)I_ec_r\Bigg]
\end{aligned} \tag{23}$$

$$\begin{aligned}
\frac{\partial \mathrm{TPI}_{11}}{\partial T} &= -\frac{1}{T^2}\Bigg[D\Bigg\{\left[1-e^{-b(T-t_s)}\right]X - c_l(T-t_s) + \frac{c_b}{b}(T-t_s)e^{-b(T-t_s)} + (1-\delta)c_rI_et_s\left(T-\frac{t_s}{2}\right)\\
&\quad + Y - \frac{1}{b}\left\{1-e^{-b(T-t_s)}\right\}i_c\delta c_pt_s + \frac{1}{\theta^2}(c_h+c_pI_c)\left\{\theta(t_s-t_d) - e^{\theta(t_s-t_d)}\right\} + c_rt_s\Bigg\}\\
&\quad + Q_0e^{\alpha M}(i_c\delta c_pt_s - Z) - A - c_aA_d\Bigg] + \frac{D}{T}\Bigg\{Xbe^{-b(T-t_s)} - c_l + \frac{c_b}{b}e^{-b(T-t_s)}\\
&\quad - c_b(T-t_s)e^{-b(T-t_s)} + (1-\delta)c_rI_et_s - i_c\delta c_pt_se^{-b(T-t_s)}\Bigg\}
\end{aligned} \tag{24}$$

$$\begin{aligned}
\frac{\partial^2 \mathrm{TPI}_{11}}{\partial T^2} &= -\frac{2D}{T^2}\Bigg\{Xbe^{-b(T-t_s)} - c_l + \frac{c_b}{b}e^{-b(T-t_s)} - c_b(T-t_s)e^{-b(T-t_s)} + (1-\delta)c_rI_et_s\\
&\quad - i_c\delta c_pt_se^{-b(T-t_s)}\Bigg\} + \frac{2}{T^3}\Bigg[D\Bigg\{\left[1-e^{-b(T-t_s)}\right]X - c_l(T-t_s) + \frac{c_b}{b}(T-t_s)e^{-b(T-t_s)}\\
&\quad + (1-\delta)c_rI_et_s\left(T-\frac{t_s}{2}\right) + Y - \frac{1}{b}\left\{1-e^{-b(T-t_s)}\right\}i_c\delta c_pt_s + \frac{1}{\theta^2}(c_h+c_pI_c)\{\theta(t_s-t_d)
\end{aligned}$$

$$-e^{\theta(t_s-t_d)}\Big\} + c_r t_s\Big\} + Q_0 e^{\alpha M}(i_c\delta c_p t_s - Z) - A - c_a A_d\Big] + \frac{D}{T}\Big\{-Xb^2 e^{-b(T-t_s)}$$
$$-2c_b e^{-b(T-t_s)} + bc_b(T-t_s)e^{-b(T-t_s)} + bi_c\delta c_p t_s e^{-b(T-t_s)}\Big\} \tag{25}$$

$$\begin{aligned}\frac{\partial^2 \mathrm{TPI}_{11}}{\partial Q_0 \partial T} &= -\frac{1}{T^2}\Big[De^{-b(T-t_s)}\Big\{-Xb - \frac{c_b}{b} + c_b(T-t_s) + \frac{1}{b}i_c\delta c_p(1+bt_s)\Big\}\frac{\partial t_s}{\partial Q_0}\\ &+\Big\{D\Big[c_l + (1-\delta)c_r I_e(T-t_s) - \frac{1}{b}i_c\delta c_p + \frac{1}{\theta}(c_h + I_c c_p)\Big(1-e^{\theta(t_s-t_d)}\Big) + c_r\Big]\\ &+Q_0 e^{\alpha M} i_c\delta c_p\Big\}\frac{\partial t_s}{\partial Q_0} + e^{\alpha M}(i_c\delta c_p t_s - Z)\Big] + \frac{D}{T}\Big[\Big\{Xb^2 e^{-b(T-t_s)} + 2c_b e^{-b(T-t_s)}\\ &-bc_b(T-t_s)e^{-b(T-t_s)} + (1-\delta)c_r I_e - bi_c\delta c_p t_s e^{-b(T-t_s)} - i_c\delta c_p e^{-b(T-t_s)}\Big\}\frac{\partial t_s}{\partial Q_0}\Big]\end{aligned} \tag{26}$$

where, $\frac{\partial t_s}{\partial Q_0} = \frac{1}{D}e^{[\alpha M - \theta(t_s-t_d)]}$ and $\frac{\partial^2 t_s}{\partial Q_0^2} = -\frac{\theta}{D^2}e^{2[\alpha M-\theta(t_s-t_d)]}$

Similarly, there exist first order and second order derivatives of TPI_{12}, TPI_{13}.

Let us assume that, $\frac{\partial^2 \mathrm{TPI}_{1i}}{\partial Q_0^2} = \triangle_{1i}$ and $\frac{\partial^2 \mathrm{TPI}_{1i}}{\partial T^2} = \triangle_{2i}$ for the ith objective function TPI_{1i}, for $i = 1,2,3$.

Now let us presume that, there exists at least one solution (Q_0^*, T^*) of the set of equations $\frac{\partial \mathrm{TPI}_{1i}}{\partial Q_0} = 0$ and $\frac{\partial \mathrm{TPI}_{1i}}{\partial T} = 0$ for a constant ${A_d}^0$ of A_d. Now, the Hessian matrix is given by:

$$H = \begin{pmatrix} \triangle_{1i} & \frac{\partial^2(\mathrm{TPI}_{1i})}{\partial Q_0 \partial T} \\ \frac{\partial^2(\mathrm{TPI}_{1i})}{\partial T \partial Q_0} & \triangle_{2i}\end{pmatrix}$$

Lemma 2. *For fixed ${A_d}^0$ and $i = 1,2,3$ if $\triangle_{1i} < 0$, $\triangle_{2i} < 0$ and $|H| > 0$, the optimum value $\mathrm{TPI}_{1i}(Q_0^*, T^*)$ exists . Otherwise, the optimum value does not exists.*

Proof. Here, $Q_0 = Q_0^*$ and $T = T^*$ are the solutions of $\frac{\partial \mathrm{TPI}_{1i}}{\partial Q_0} = 0$ and $\frac{\partial TPI_{1i}}{\partial T} = 0$ respectively for $i = 1,2,3$.

Now, if $\triangle_{1i} < 0$, then $\frac{\partial^2 \mathrm{TPI}_{1i}}{\partial Q_0^2} < 0$ at (Q_0^*, T^*) for fixed ${A_d}^0$ and $i = 1,2,3$.

Again, if $\triangle_{2i} < 0$, then $\frac{\partial^2 \mathrm{TPI}_{1i}}{\partial T^2} < 0$ at (Q_0^*, T^*) for fixed ${A_d}^0$ and $i = 1,2,3$.

Also, if $|H| > 0$, then $\triangle_{1i} \cdot \triangle_{2i} - \frac{\partial^2 \mathrm{TPI}_{1i}}{\partial Q_0 \partial T} \cdot \frac{\partial^2 \mathrm{TPI}_{1i}}{\partial T \partial Q_0} > 0$ at (Q_0^*, T^*) for fixed ${A_d}^0$ and $i = 1,2,3$.

Therefore, all condition have been satisfied for the existence of optimum solution of $\mathrm{TPI}_{1i}(Q_0, T)$ at $Q_0 = Q_0^*$ and $T = T^*$ for $i = 1,2,3$.

Hence, for fixed ${A_d}^0$ and $i = 1,2,3$, the optimum value $\mathrm{TPI}_{1i}(Q_0^*, T^*)$ exists if $\triangle_{1i} < 0$, $\triangle_{2i} < 0$ and $|H| > 0$.

Hence the proof. □

Algorithm. Through the following steps we have given out the solution structure of our problem. Here, TPI_{1i} denotes the expected integrated profit function of different cases depending on the different length of the credit period M, where $i = 1,2,3$.

Step 1: At first define the domain of A_d. Suppose, it is $[{A_d}^0, A_d^{\max}]$, *i.e.*, A_d takes an integer value for $[{A_d}^0, A_d^{\max}]$.
Step 2: Assign the initial value of the profit function as $\mathrm{TPI}_{1i}^0 = 0$ and $A_d = {A_d}^0$.
Step 3: Compute $TPI_{1i}(Q_0, T)$ for determined A_d.
Step 4: Calculate the first order derivative $\frac{\partial \mathrm{TPI}_{1i}}{\partial Q_0}$ and $\frac{\partial \mathrm{TPI}_{1i}}{\partial T}$.
Step 5: Find out a positive root of the equations $\frac{\partial \mathrm{TPI}_{1i}}{\partial Q_0} = 0$ and $\frac{\partial \mathrm{TPI}_{1i}}{\partial T} = 0$ and let it is $Q_0 = Q_0^*$ and $T = T^*$.
Step 6: we have to calculate $\triangle_{1i}$ and $\triangle_{2i}$.

Step 7: Calculate the hessian matrix:

$$H = \begin{pmatrix} \triangle_{1i} & \frac{\partial^2(\mathrm{TPI}_{1i})}{\partial Q_0 \partial T} \\ \frac{\partial^2(\mathrm{TPI}_{1i})}{\partial T \partial Q_0} & \triangle_{2i} \end{pmatrix}$$

Step 8: If $\triangle_{1i} < 0$, $\triangle_{2i} < 0$ and $|H| > 0$ at (Q_0^*, T^*), then Q_0^*, T^* and TPI_{1i}^* are the optimum solutions.

Step 9: If $\mathrm{TPI}_{1i}^* > \mathrm{TPI}_{1i}^0$, then set $\mathrm{TPI}_{1i}^0 = \mathrm{TPI}_{1i}^*$, $Q_0^{**} = Q_0^*$, $T^{**} = T^*$, $A_d^{**} = A_d$ and go to next step, otherwise goto step 12.

Step 10: Set $A_d = A_d + 1$.

Step 11: If $A_d \leq A_d^{\max}$, then goto step 3, otherwise goto next step.

Step 12: Print the optimum values of Q_0, A_d, T and ITP_{1i} as $Q_0^{**}, A_d^{**}, T^{**}$ and TPI_{1i}^0 respectively.

6. Numerical analysis and sensitivity analysis

6.1. Numerical example:

In this section, we illustrate numerical examples to study the feasibility of the proposed model. To explain our model numerically, the following example has been considered where the values of parameters of the model have been seemed to be realistic.

Example 1 (When the supplier offers a partial credit policy). A local retail store is managed by a retailer and he/she sells various commodities, such as fruits, vegetables, etc., to his/her customers. These items are supplied to the retail store by a supplier and the supplier purchases the Q amount of items from a farmer at a price of \$10 per unit and he/she delivers rapidly to the retail store at the cost of \$20 per unit. The retailer received the items as soon as the order was placed and the retailer's ordering cost \$200 per order. All of these products are non-instantaneous deteriorating and the product will deteriorate at a constant rate of 0.65. So, the item remains fresh till the time 0.5 years. To increase the sales rate, the supplier decides to offer a credit period M to the retailer, where the retailer has paid 0.5 portion of his/her total purchase cost at the purchasing time and the remaining will be paid the credit period onwards. The retailer takes a bank loan to pay this portion with an interest rate 0.07. Due to this offered credit period, the supplier faces an opportunity loss at the rate of \$0.065. The supplier charges interest to the retailer at the rate of \$0.14 due to late payment and the retailer earns interest at the rate of \$0.12 due to the credit period. After receiving the credit period M, the retailer orders the products depending on the length of this offered credit period. Here, the retail store uses an advertisement policy to promote its products. He/she can use a maximum of 50 advertisements with the corresponding advertisement cost is \$85 per advertisement. Due to this advertisement, the maximum demand rate will be 2 times. The retailer sells his/her products with markup and decides when he/she uses no advertising, his/her markup for selling price is 1.1 but for the maximum given advertisement, his/her markup due to advertisement is 0.03. The retailer faced a normal market demand D_0, which is stochastic. The holding cost for the retailer is \$2.5 per unit. Due to this stochastic demand and also for deterioration, the retailer faces shortages and after that, some demands are partially backlogged and the backordered cost of \$3.2 per unit. Also, the retailer faces a lost sale and the cost of lost sale are \$4 per unit. Now, find out the optimal integrated profit, optimal profit of supplier and retailer individually, along with the retailer's optimal ordering quantity, the optimal number of advertisements, optimal shortage time with optimal shortages and the optimal length of the inventory cycle.

Solution: For the stochastic component, we consider the continuous uniform distribution demand rate

$$f(D_0) = \begin{cases} \dfrac{1}{c-a}; & \text{when} \quad a \leq D_0 \leq c \\ 0; & \text{otherwise} \end{cases}$$

where the minimum amount of demand is 200 units/year and the maximum amount of demand is 900 units/year.

TABLE 2. Optimal results for different cases of M.

Case	Q_0	S	Q	A_d	t_s	T	TPS	TPR	TPI
$0 < M \leq t_d$	187.480	103.118	371.839	22	1.132	1.720	2105.673	1274.211	3379.884
$t_d < M \leq t_s$	161.452	101.552	378.606	22	1.159	1.736	2095.515	1309.204	3404.719
$t_s < M \leq T$	73.050	102.432	410.754	22	1.275	1.870	1967.489	1365.076	3332.565

The values of the parameters with their suitable units are
$a = 200$, $c = 900$, $p = 10$, $b = 0.7$, $c_p = 20$, $c_h = 2.5$, $c_a = 85$, $c_b = 3.2$, $c_l = 4.0$, $A_m = 50$, $A = 200$, $m_1 = 1.1$, $t_d = 0.5$, $i_c = 0.07$, $I_c = 0.14$, $I_e = 0.12$, $I_l = 0.065$, $k = 0.4$, $\alpha = 0.9$, $\beta = 12$, $\gamma = 0.03$, $\delta = 0.5$, $\theta = 0.55$, $\eta = 2$
The different values of M for different cases are:

Case 1. When $0 < M \leq t_d$: $M = 0.4$.
Case 2. When $t_d < M \leq t_s$: $M = 0.6$.
Case 3. When $t_s < M \leq T$: $M = 1.6$.

From Table 2, when the supplier offers credit period M less than or equal to the time at which the deterioration starts t_d, *i.e.*, $M \leq t_d$, then the retailer will order $Q = 371.839$ quantity, whereas, when $t_d < M \leq t_s$, *i.e.*, when the credit period M greater than t_d but less than or equal to the shortage time t_s, then the retailer will order $Q = 378.606$ quantity and when the credit period M greater than the shortage time t_s then the retailer will order maximum quantity as $Q = 410.754$ compared to other subcases. That means the retailer will order the maximum quantity for $M > t_s$ because the retailer will face a shortage before the end of credit time, so needed to order more quantity compared to other cases.

From Table 2, it is clear that the expected profit of the supplier (TPS) is maximum for case 1 *i.e.*, for $0 < M \leq t_d$ because the smaller length of M gives a lower opportunity loss for the supplier. Also, from Table 2, we see that the expected profit of the retailer (TPR) is maximum for case 3 *i.e.*, for $t_s < M \leq T$. But from Table 2, we see that the total integrated profit of the whole system (TPI) is more profitable for case 1 *i.e.*, for $t_d < M \leq t_s$ thus, the supplier and retailer both follow case 2 to maximize the total expected integrated profit of the whole system.

Also from Table 2, we can see that for the case $t_s < M \leq T$, the total profit of the supplier is much less than the other two subcases. Because a longer credit period gives the supplier more interest loss. So, the supplier will never want to give more credit period instead of his loss. So, here we analyze sensitivity for the first two cases.

The graph in Figure 3 indicates the concavity of the integrated profit function with respect to A_d and T for fixed value of Q_0. Figure 4 indicates the concavity of the integrated profit function with respect to Q_0 and T for fixed value of A_d and Figure 5 indicates the concavity of the integrated profit function with respect to Q_0 and A_d for fixed value of T.

6.2. Sensitivity analysis

To illustrate numerically, we study the sensitivity analysis of the key parameters η, k, M, β, θ, b, δ in Tables 3–10 and discuss the effect of changes in parameter values on the optimal solution.

- From Tables 3–4 we see that, when η is increased, the retailers initial ordering quantity (Q_0), shortages (S) are increased. So, the total ordering quantity (Q) is increased. The retailers will always want to advertise more for more items. So, the number of advertisement (A_d) increases with the increasing value of η. As more items are ordered, the shortage time (t_s), length of cycle (T), retailers total profit (TPR) increases automatically and the suppliers total profit (TPS) decreases and slightly increases for $\eta > 2$. The integrated total profit (TPI) always increases with the increasing value of η, which is described in Figure 6.
- From Tables 3–4 and Figure 7, the increasing value of k increases the demand structure, so it always increases the retailer's ordering quantity Q. Therefore, when k is increased, then Q_0, S, Q increases and also A_d is increased. When the retailer orders more quantity, then obviously, he/she will take more time to sell all

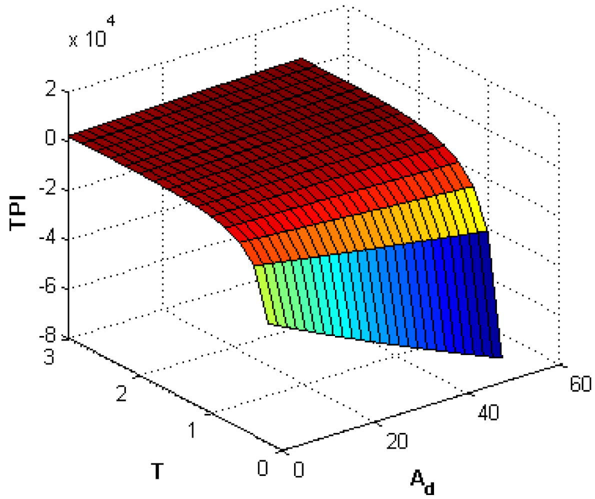


FIGURE 3. Integrated profit function with respect to A_d and T for fixed Q_0.

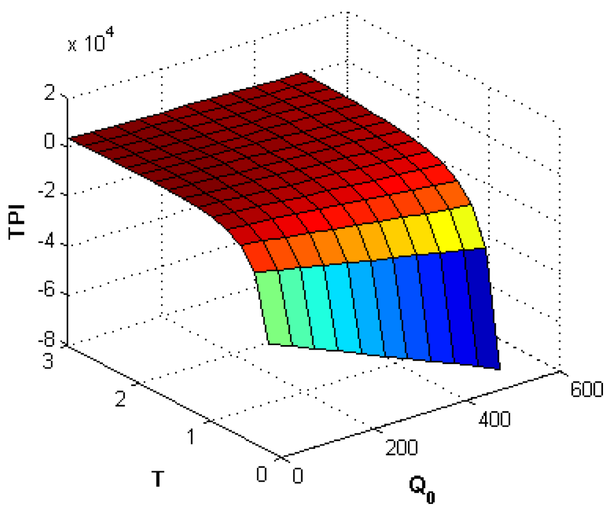


FIGURE 4. Integrated profit function with respect to Q_0 and T for fixed A_d,

those items. So, increasing value of k always increases t_s, T, then TPR automatically increases, while the supplier's total profit (TPS) slightly decreases and increases for $k > 0.5$. The integrated total profit (TPI) increases with the increasing value of k.

– From Tables 5–6 and Figure 8, see that increasing value of credit period (M), the retailer's ordering quantity (Q) automatically increased. So, the shortage time (t_s), length of cycle (T), retailer's total profit (TPR) are increased due to sell more items. Due to the retailer's ordering quantity begin higher, the supplier's profit (TPS) increases at first, but as the increasing credit period increases the rate of opportunity loss, the supplier profit starts to decrease and A_d remains unchanged. Also, total integrated profit (TPI) increases for the case $0 < M \leq t_d$ and decreases for the case $t_d < M \leq t_s$.
– Next Tables 7–8 and Figure 9 it is observed that, the higher demand sensitivity of customer about retail price (β) reduces retailer's initial ordering quantity (Q_0), shortages (S), total ordering quantity (Q) and due to less quantity the number of advertisement (A_d), the shortage time (t_s), length of cycle (T) are decreases.

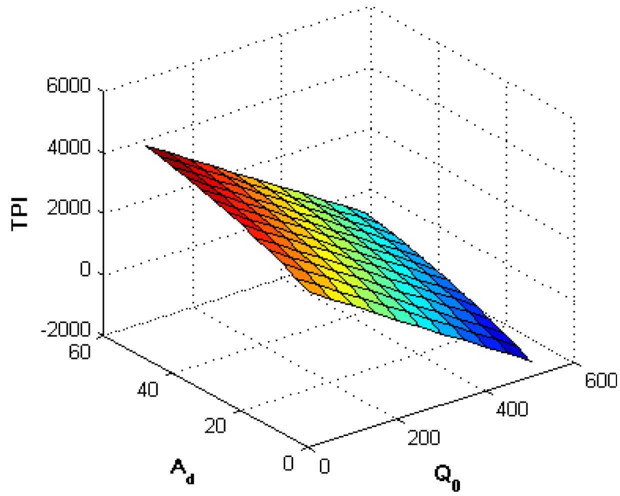


FIGURE 5. Integrated profit function with respect to Q_0 and A_d for fixed T.

As a result, the lower values of Q decreases the supplier's total profit (TPS), retailer's total profit (TPR), integrated total profit (TPI).

- From Tables 7 to 8 we observed that, the increasing values of product deterioration rate θ decreases the retailer's ordering quantity Q. Due to this shortage time t_s and cycle time T are decreases. So, here retailer's total profit (TPR), integrated total profit (TPI) decreases gradually for both the cases $0 < M \leq t_d$ and $t_d < M \leq t_s$. But, the shortage S and the supplier's total profit (TPS) fluctuated when the value of θ raises more, which is describe in Figure 10.
- Tables 7–8 and Figure 11 showing that when the value of the backlogging parameter (b) increases, the retailer's initial ordering quantity (Q_0) increases but the shortages (S) and the retailer's total ordering quantity (Q) are decreases. Due to retailer's having low stock items, the stock tends to run out quickly. For that, the shortage time (t_s) and length of cycle (T) decreases with the increasing value of b. As a result retailer's total profit (TPR), integrated total profit (TPI) are decrease when b increases. But the supplier total profit (TPS) increase and A_d remain almost the same for grater values of b.
- From Tables 9–10 we have studied the outcome of δ on profit structure. We seen that, when fraction of paid purchase cost at ordering time (δ) increases, then it decreases the retailer's demand. So, with the increasing values of δ, decreases the retailer's initial ordering quantity (Q_0), the shortages (S) and the retailer's total ordering quantity (Q). The shortage time (t_s), cycle time (T) and number of advertisements (A_d) will also be decreases for lower quantities. As a result, retailer's total profit (TPR), integrated total profit (TPI) are decreases. As the δ portion of the retailer's purchase cost is paid at the ordering time, the opportunity loss of the supplier is much less, so as the δ increases, the supplier's profit (TPS) also increases, which is describe in Figure 12.

Example 2 (**When the supplier offers a full credit policy).** In this example, the supplier gives a full credit period offered to the retailer (*i.e.*, $\delta = 0$). So there is no need for the retailer to take loan from bank (*i.e.*, $i_c = 0$) and considering the remaining parameters same as Example 1.

Solution: The values of the parameters with their suitable units are

$a = 200$, $c = 900$, $p = 10$, $b = 0.7$, $c_p = 20$, $c_h = 2.5$, $c_a = 85$, $c_b = 3.2$, $c_l = 4.0$, $A_m = 50$, $A = 200$, $m_1 = 1.1$, $t_d = 0.5$, $I_c = 0.14$, $I_e = 0.12$, $I_l = 0.065$, $k = 0.4$, $\alpha = 0.9$, $\beta = 12$, $\gamma = 0.03$, $\theta = 0.55$, $\eta = 2$

The different values of M for different cases are:

Case 1. When $0 < M \leq t_d$: $M = 0.4$.

Table 3. Sensitivity of parameter η, k for case $0 < M \leq t_d$.

Parameter	Values	Q_0	S	Q	A_d	t_s	T	TPS	TPR	TPI
η	1.50	136.807	78.681	274.771	7	0.762	1.117	2396.507	475.419	2871.926
	1.75	161.561	94.114	325.684	14	0.951	1.436	2208.325	844.464	3052.788
	2.00	187.480	103.118	371.839	22	1.132	1.720	2105.673	1274.211	3379.884
	2.25	212.279	108.710	412.976	29	1.258	1.896	2121.420	1745.337	3866.757
	2.40	227.692	111.610	437.968	33	1.317	1.970	2164.796	2073.309	4238.106
k	0.23	143.959	84.171	290.512	9	0.821	1.218	2322.600	577.152	2899.752
	0.30	158.458	92.598	319.720	13	0.926	1.394	2234.094	796.255	3030.349
	0.40	187.480	103.118	371.839	22	1.132	1.720	2105.673	1274.211	3379.884
	0.50	224.121	110.326	431.566	33	1.330	1.997	2105.061	1972.466	4077.527
	0.60	283.238	117.202	523.176	50	1.559	2.268	2246.878	3341.744	5588.622

Table 4. Sensitivity of parameter η, k for case $t_d < M \leq t_s$.

Parameter	Values	Q_0	S	Q	A_d	t_s	T	TPS	TPR	TPI
η	1.50	122.537	80.040	290.315	8	0.825	1.195	2334.291	544.391	2878.683
	1.75	142.007	93.693	337.378	15	1.007	1.501	2159.935	910.775	3070.711
	2.00	161.452	101.552	378.606	22	1.159	1.736	2095.515	1309.204	3404.719
	2.25	182.225	107.167	419.867	29	1.284	1.910	2111.603	1785.615	3897.218
	2.40	195.211	110.069	445.052	33	1.343	1.984	2154.848	2116.881	4271.730
k	0.23	125.669	82.697	298.347	9	0.852	1.242	2308.144	600.827	2908.971
	0.30	139.537	92.334	331.781	14	0.982	1.460	2183.787	863.031	3046.818
	0.40	161.452	101.552	378.606	22	1.159	1.736	2095.515	1309.204	3404.719
	0.50	192.108	108.821	438.480	33	1.355	2.011	2095.512	2015.000	4110.512
	0.60	241.905	115.766	530.876	50	1.583	2.281	2236.725	3396.050	5632.775

Table 5. Sensitivity of parameter M for case $0 < M \leq t_d$.

Parameter	Values	Q_0	S	Q	A_d	t_s	T	TPS	TPR	TPI
M	0.1	233.620	106.326	361.947	21	1.074	1.672	2151.019	1167.885	3318.905
	0.2	216.039	104.951	363.598	21	1.084	1.672	2145.996	1196.400	3342.395
	0.3	200.024	103.740	365.765	21	1.095	1.675	2140.922	1221.761	3362.683
	0.4	187.480	103.118	371.839	22	1.132	1.720	2105.673	1274.211	3379.884
	0.5	173.928	102.246	375.020	22	1.145	1.727	2100.727	1293.416	3394.143

Case 2. When $t_d < M \leq t_s$: $M = 0.6$.
Case 3. When $t_s < M \leq T$: $M = 1.6$.

From Table 11, we see that with the increase in retailer's ordering quantity (Q), suppliers expected profit (TPS), retailer expected profit (TPR) and total integrated profit of the whole system (TPI) is the same as in Table 2.

But compared to full and partial credit, we see from Tables 2 to 11 that the retailer will order more quantity in the full credit for any case. So, the retailer's expected profit (TPR) is more during the full credit than in the partial credit.

TABLE 6. Sensitivity of parameter M for case $t_d < M \leq t_s$.

Parameter	Values	Q_0	S	Q	A_d	t_s	T	TPS	TPR	TPI
	0.55	167.572	101.874	376.776	22	1.152	1.731	2098.188	1301.785	3399.973
	0.60	161.452	101.552	378.606	22	1.159	1.736	2095.515	1309.204	3404.719
M	0.70	149.885	101.056	382.481	22	1.173	1.747	2089.779	1321.295	3411.074
	0.80	139.154	100.746	386.629	22	1.188	1.759	2083.546	1329.890	3413.436
	1.00	119.946	100.649	395.667	22	1.217	1.787	2069.677	1337.388	3407.065

TABLE 7. Sensitivity of parameter $\beta,\ \theta,\ b$ for case $0 < M \leq t_d$.

Parameter	Values	Q_0	S	Q	A_d	t_s	T	TPS	TPR	TPI
	9	316.708	127.710	581.657	50	1.458	2.066	2741.392	4600.056	7341.448
	10	270.047	119.373	506.439	42	1.411	2.057	2398.338	3146.317	5544.655
β	11	224.865	111.978	434.284	31	1.272	1.899	2226.590	2041.931	4268.521
	12	187.480	103.118	371.839	22	1.132	1.720	2105.673	1274.211	3379.884
	13	151.567	90.033	307.277	13	0.947	1.440	2078.516	678.889	2257.406
	0.42	201.229	101.766	390.193	22	1.225	1.804	2106.994	1334.925	3441.920
θ	0.46	196.626	102.201	384.031	22	1.194	1.775	2107.147	1314.235	3421.382
	0.50	192.371	102.619	378.350	22	1.165	1.749	2106.762	1295.426	3402.188
	0.55	187.480	103.118	371.839	22	1.132	1.720	2105.673	1274.211	3379.884
	0.58	182.754	102.969	364.916	21	1.090	1.665	2134.908	1232.960	3367.868
	0.62	186.186	112.573	379.438	22	1.126	1.763	2096.553	1321.318	3417.871
	0.66	186.854	107.632	375.456	22	1.129	1.740	2101.143	1297.059	3398.202
b	0.70	187.480	103.118	371.839	22	1.132	1.720	2105.673	1274.211	3379.884
	0.74	185.995	98.580	365.168	21	1.110	1.662	2140.139	1222.977	3363.116
	0.78	186536	94.790	362.158	21	1.113	1.645	2144.374	1203.132	3347.506

TABLE 8. Sensitivity of parameter $\beta,\ \theta,\ b$ for case $t_d < M \leq t_s$.

Parameter	Values	Q_0	S	Q	A_d	t_s	T	TPS	TPR	TPI
	9	270.827	126.005	590.747	50	1.482	2.081	2727.970	4664.792	7392.763
	10	231.114	117.775	514.369	42	1.436	2.071	2387.069	3199.573	5586.641
β	11	192.963	110.386	441.511	31	1.298	1.915	2216.151	2085.254	4301.405
	12	161.452	101.552	378.606	22	1.159	1.736	2095.515	1309.204	3404.719
	13	133.453	89.819	318.824	14	1.004	1.506	2033.217	740.450	2773.667
	0.42	175.025	100.365	400.709	23	1.284	1.864	2065.122	1405.840	3470.962
	0.46	169.271	100.542	291.012	22	1.223	1.792	2096.092	1352.677	3448.769
θ	0.50	165.637	101.003	385.237	22	1.193	1.766	2096.139	1332.274	3428.413
	0.55	161.452	101.552	378.606	22	1.159	1.736	2095.515	1309.204	3404.719
	0.58	159.111	101.870	374.905	22	1.141	1.720	2094.838	1296.511	3391.348
	0.62	160.429	110.909	386.206	22	1.154	1.779	2086.274	1354.710	3440.985
	0.66	160.957	106.019	382.223	22	1.157	1.757	2090.934	1331.267	3422.201
b	0.70	161.452	101.552	378.606	22	1.159	1.736	2095.515	1309.204	3404.719
	0.74	161.917	97.456	375.306	22	1.162	1.717	2100.004	1288.400	3388.404
	0.78	160.673	93.295	369.012	21	1.140	1.662	2133.972	1239.190	3373.162

TABLE 9. Sensitivity of parameter δ for case $0 < M \leq t_d$.

Parameter	Values	Q_0	S	Q	A_d	t_s	T	TPS	TPR	TPI
	0.2	221.226	110.610	427.701	23	1.302	1.958	2093.639	1582.781	3676.419
	0.3	209.395	107.845	407.978	23	1.249	1.885	2086.023	1485.927	3571.950
δ	0.4	195.765	104.927	385.523	22	1.717	1.771	2109.389	1354.569	3463.958
	0.5	187.480	103.118	371.839	22	1.132	1.720	2105.673	1274.211	3379.884
	0.6	177.218	100.968	354.979	21	1.068	1.629	2133.279	1158.624	3291.903

TABLE 10. Sensitivity of parameter δ for case $t_d < M \leq t_s$.

Parameter	Values	Q_0	S	Q	A_d	t_s	T	TPS	TPR	TPI
	0.2	190.001	109.507	435.549	23	1.329	1.977	2065.840	1625.740	3691.581
	0.3	179.959	106.588	415.399	23	1.276	1.902	2064.320	1526.238	3590.558
δ	0.4	169.307	103.723	394.254	22	1.203	1.795	2093.772	1401.077	3494.849
	0.5	161.452	101.552	378.606	22	1.159	1.736	2095.515	1309.204	3404.719
	0.6	152.832	99.244	361.505	21	1.096	1.645	2128.727	1190.832	3319.559

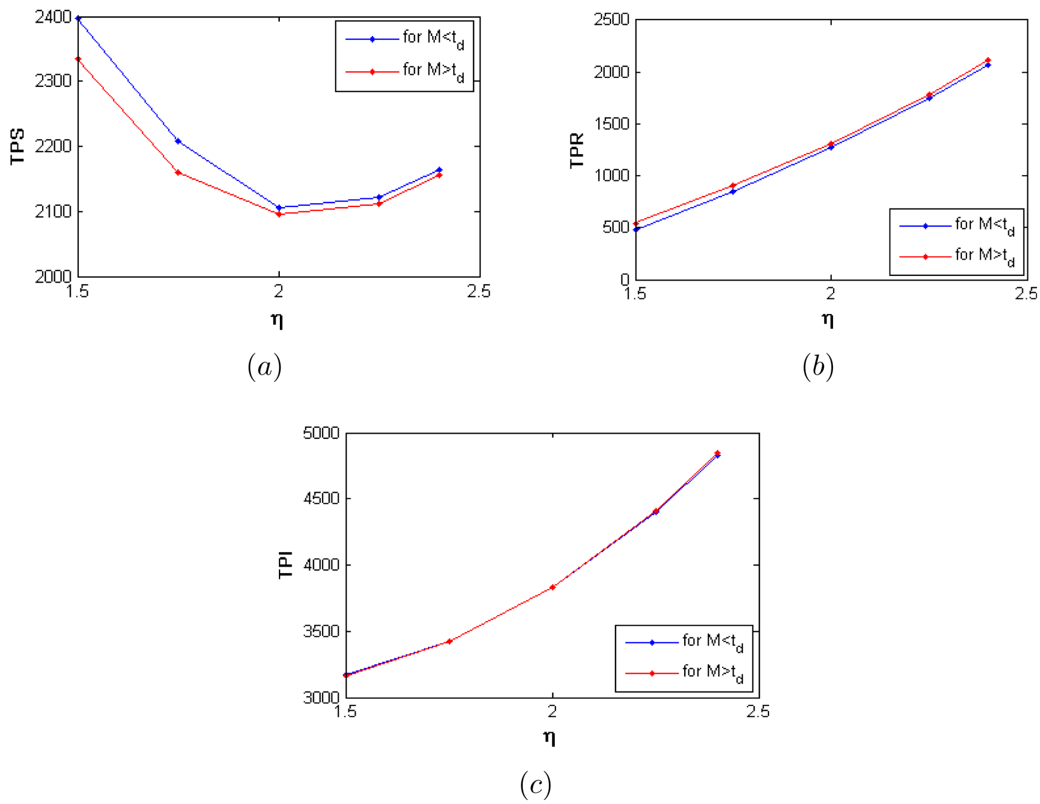
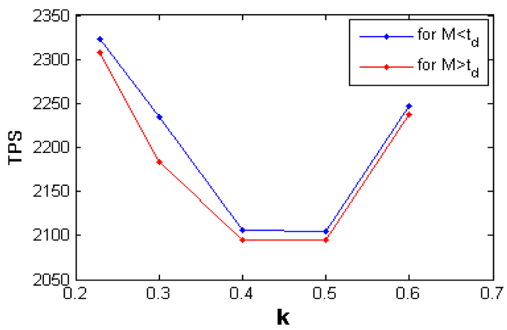
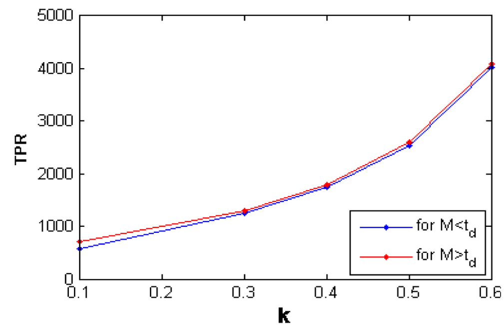


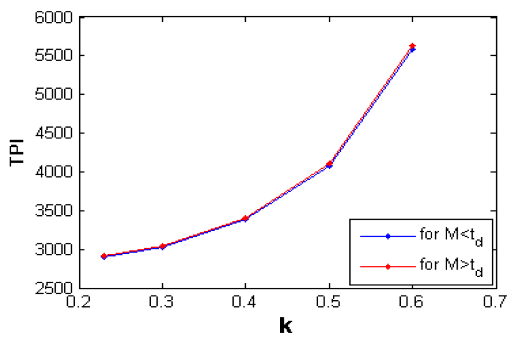
FIGURE 6. (a) η *vs.* TPS, (b) η *vs.* TPR and (c) η *vs.* TPI.



(a)

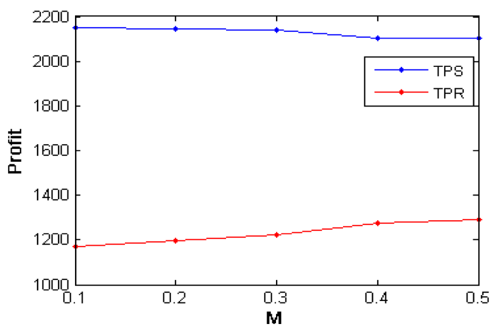


(b)

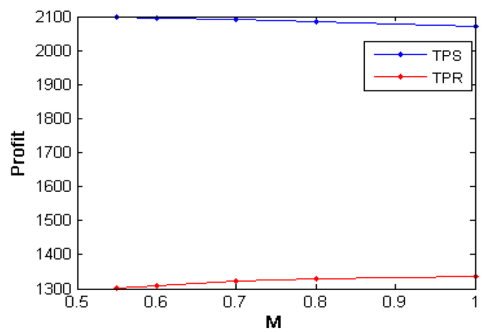


(c)

FIGURE 7. (a) k *vs.* TPS, (b) k *vs.* TPR and (c) k *vs.* TPI.



When $0 < M \leq t_d$



When $t_d < M \leq t_s$

FIGURE 8. M *vs.* Profit.

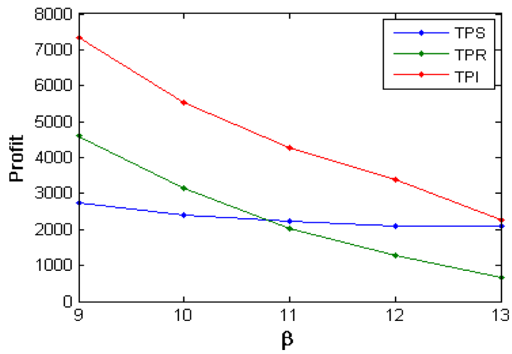
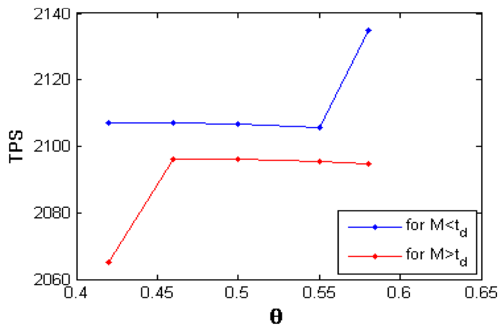
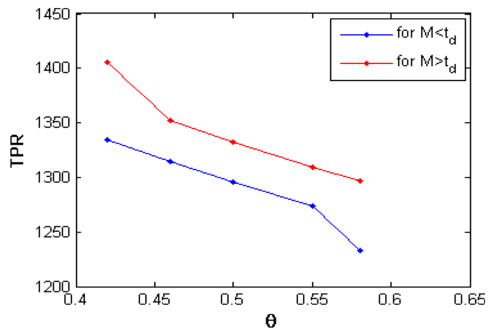


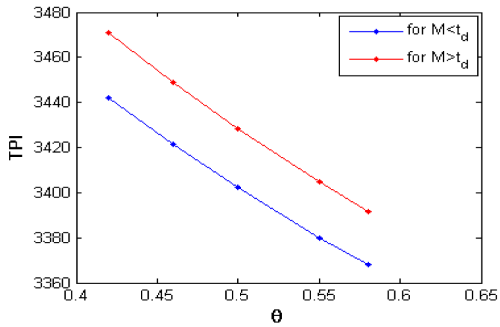
FIGURE 9. β *vs.* Profit.



(a)

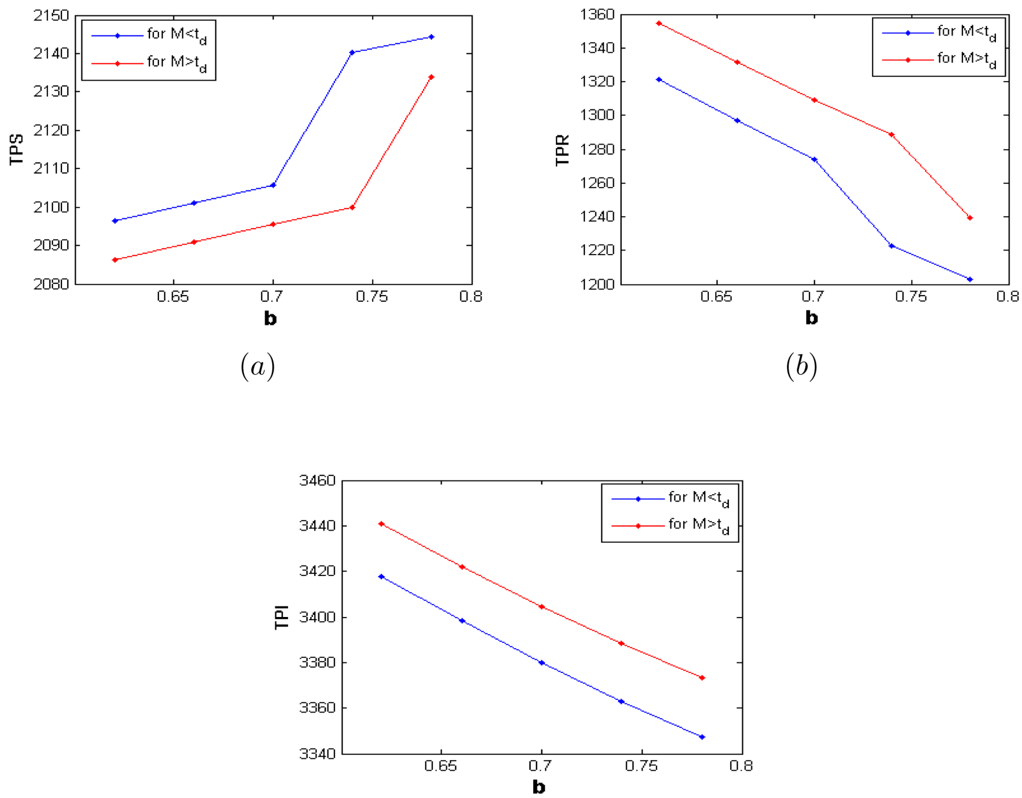


(b)



(c)

FIGURE 10. (a) θ *vs.* TPS, (b) θ *vs.* TPR and (c) θ *vs.* TPI.



(a) (b)

(c)

FIGURE 11. (a) b *vs.* TPS, (b) b *vs.* TPR and (c) b *vs.* TPI.

TABLE 11. Optimal results for different cases of M.

Case	Q_0	S	Q	A_d	t_s	T	TPS	TPR	TPI
$0 < M \leq t_d$	253.101	117.849	480.626	24	1.455	2.183	2087.054	1818.060	3905.114
$t_d < M \leq t_s$	219.329	117.241	493.610	25	1.512	20254	2019.534	1895.104	3914.638
$t_s < M \leq T$	94.871	119.224	519.647	26	1.600	2.376	1731.975	2034.875	3766.850

Also, from Tables 2 to 11, we see that the supplier can earn more profit in the partial credit policy compared to the full credit policy. But the total integrated profit of the whole system (TPI) is maximum in the full credit policy.

6.3. Sensitivity analysis of markup:

Markup is a very important parameter of the profit structure of the whole system. If the retailer increases the markup too much for his profit, it negatively impacts demand. Customers always have a tendency to procure the products at lower prices and due to the higher retail price, the customers will never be interested in buying the products. Here, we discuss the effect of changes in parameter values on the optimal solution.

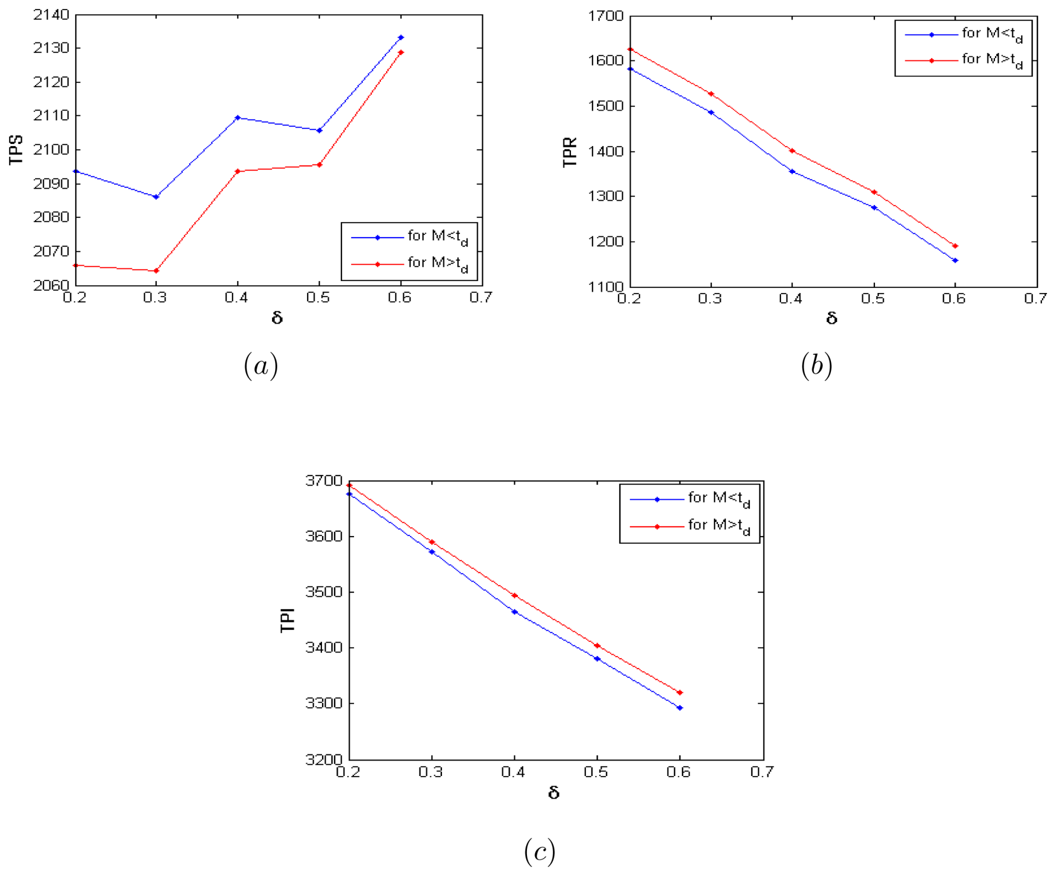


FIGURE 12. (a) δ *vs.* TPS, (b) δ *vs.* TPR and (c) δ *vs.* TPI.

For full credit:

Since from Figure 13 and Table 12, the increasing value of m_1 decreases the retailers ordering quantity Q. Since the increasing value of m_1 increases the retailer's selling price. So it will be a negative impact on customer demand. So, when the customer's demand decreases, the retailer obviously orders less quantity. As a result, due to increasing the retailer's selling price, retailer's total profit (TPR) first increases and then for lower customers demand, the retailer's total profit decreases gradually. The supplier's total profit (TPS) also decreases with the increasing value of this markup.

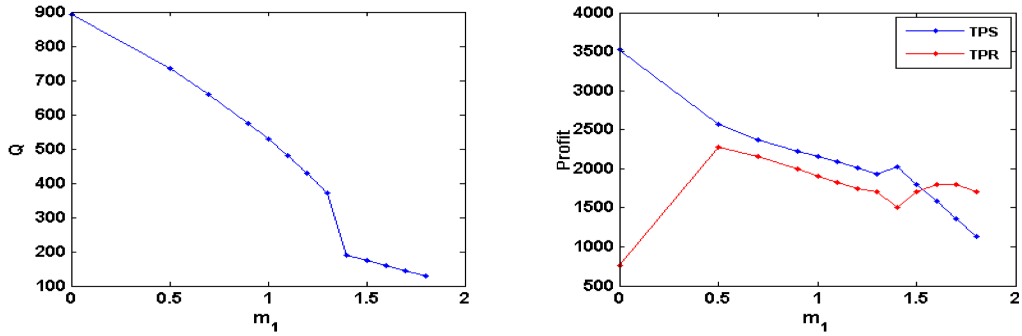
From Table 12, the increasing value of m_1 reduces the number of advertisements (A_d) and after a certain time, $A_d = 0$. Because, when increasing the value of m_1, increases A_d, total markup (m) increases too much. As a result, retailer's selling price will increase a lot, which will be negative impact on customer demand. So, when m_1 is increased, then integrated total profit (TPI) first increases and then it tends to decreases, which is described in Figure 14.

But, from Figure 14, we see that when we fixed the number of advertisements A_d, if m_1 increased, then m increased. So, though the retailer's profit grows at first, the demand reduction will continue to decrease profit after a specific value of m_1.

Example 3 (When their is no credit policy). Here, the suppler does not offer any credit policy to its retailer, so the retailer has to pay the entire purchase cost at the time of purchase (*i.e.*, $\delta = 1$ and $M = 0$). Considering all other parametric values are the same as in Example 1.

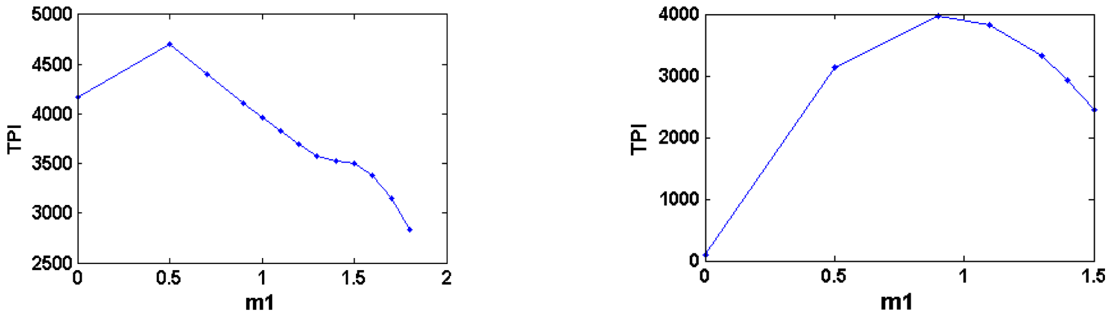
TABLE 12. Sensitivity of parameter m_1 for $0 < M \leq t_d$.

Parameter	Values	Q_0	S	Q	A_d	m	D	t_s	T	TPS	TPR	TPI
	0	445.539	253.849	892.454	50	1.50	365.729	1.449	2.399	3526.024	762.133	4288.157
	0.5	391.983	173.787	735.628	50	2.00	245.729	1.744	2.720	2563.422	2277.640	4841.062
	0.7	351.081	155.733	658.947	43	1.99	227.365	1.707	2.640	2366.465	2163.262	4529.727
	0.9	304.859	138.249	575.213	34	1.89	216.041	1.606	2.455	2221.324	1991.717	4213.040
	1.0	279.523	128.629	529.277	29	1.87	211.628	1.534	2.326	2157.169	1900.161	4057.330
	1.1	253.101	117.849	480.626	24	1.82	206.580	1.455	2.183	2087.054	1818.060	3905.114
	1.2	225.340	105.653	428.640	19	1.77	200.825	1.365	2.021	2010.131	1748.959	3759.089
m_1	1.3	195.814	91.649	372.315	14	1.72	194.281	1.261	1.833	1925.217	1698.280	3623.497
	1.4	104.447	39.026	189.306	0	1.40	214.000	0.692	0.887	2023.553	1511.221	3534.774
	1.5	97.416	35.361	174.991	0	1.50	190.000	0.721	0.920	1802.551	1705.121	3507.672
	1.6	89.596	31.913	160.330	0	1.60	166.000	0.755	0.961	1581.052	1801.707	3382.759
	1.7	81.322	28.601	145.162	0	1.70	142.000	0.795	1.013	1359.021	1801.106	3160.127
	1.8	72.505	25.346	129.270	0	1.80	118.000	0.846	1.078	1136.346	1703.666	2840.012



(a) (b)

FIGURE 13. (a) m_1 *vs.* Q and (b) m_1 *vs.* Profit.



when A_d variable when A_d fixed

FIGURE 14. m_1 *vs.* TPI.

Solution: The optimum solution in this case is given by:

$Q_0 = 238.575$, $S = 91.494$, $Q = 330.069$, $A_d = 20$, $t_s = 1.002$, $T = 1.490$, TPS = \$2214.608, TPR = \$835.653 and TPI = \$3050.261

Comparing of Example 3 with Example 1 we reveals that, when the supplier does not offer any credit policy, the retailer's initial ordering quantity increases compared to Example 1, but the retailer's total ordering quantity are less than that of Example 1. So, the shortage time and length of the cycle are decreases than in Example 1. As there is no opportunity loss due to credit period, the supplier's profit increases compared to Example 1 but the retailer's profit is decreases a lot compared Example 1 and also the total integrated profit of the whole system is decreases.

6.4. Comparative study among Examples 1–3

- From Examples 1–3 it is observe that, when the supplier offers to the retailer a full or partial any credit policy, the total ordering quantity of the retailer is more than when he/she offers no credit policy. So, the retailer's profit is better in Examples 1 and 2 than in Example 3. Also, the retailer has to give more advertisements to sell more products. So, the number of advertisement is more in Examples 1 and 2 than Example 3.
- When the supplier gives a full credit policy to the retailer, the supplier's profit is much lower than in Examples 1 and 3, due to the higher opportunity loss. Since no payment is required at the time of purchase due to full credit, the retailer orders the largest amount of items in this case. So, the retailer's total profit is highest for the Example 2 and also the total integrated profit is highest in this example compared to the other two examples.
- When the supplier gives no credit policy, the retailer's total ordering quantity are much less than in Examples 1 and 2. So, among the above cases, length of inventory cycle is lowest in Example 3. Hence, the retailer's profit in Example 3 is very less compared to Examples 1 and 2. But the supplier's total profit is highest in Example 3, because of no opportunity loss.

7. Managerial implication

This model has various managerial implications.

- In reality, it is not always possible to accurately determine the demand for all things and controlling the uncertain demand is very difficult for any seller. Because of the uncertain demand, the retailer could not purchase the item for sale as per the demand. As a result, the chance of facing shortages increases. This shortage increases further when he/she decides to provide the advertisement for his/her items. The reason that the advertising may highlight the various types of information about products such as the quality, durability, benefits, price etc. Also, it builds customer awareness of the product. Thus, the advertisement attracts the customers to purchase more quantity and in that situation, sometimes the retailer is not able to fulfil their demand. As a result, the retailer faces losses. In these circumstances, our proposed model helps the retailer to determine the maximum order quantity to meet customers these stochastic demand so that he can overcome this shortage situation without facing further loss.
- Advertising is a widely used marketing technique in modern times whose main objective is to inform customers about the qualities and characteristics of products and services, mainly through newspapers, television, graffiti, handbills, posters etc. The advertising increases the customer's interest in buying the product. But, as the number of advertisements increases, the cost of the retailer also increases and if the retailer increases the selling price to maintain his profit, the customer is discouraged from buying the product for the high price. Again, the continuous advertisement makes the customers monotonous. This model helps the retailer to determine the maximum number of advertisements for the items in stock, thereby maintaining his profit and also helping to see how the changes in the customer's demand structure are due to advertisements.
- This model helps the supplier when the supplier things to provide a credit period to the retailer. The credit period is always a valuable strategy in business. But this credit period also has some advantages and

disadvantages. However, a long credit period brings to increased sales and creates a stronger relationship with the retailer, but it also increases the supplier's chance of increasing interest loss. In this model, we have discussed full and partial both credit periods and from Tables 2 and 9, we see that a partial credit period is more profitable for the supplier than a full credit period, whereas from the perspective of the retailer as well as the whole system full credit period is more profitable. Additionally, in the case of partial credit periods, the more portion of the purchase cost paid at the ordering time, the more profit the supplier will make, but the retailer will not always be able to order the items by paying that much portion and so the ordering quantity of the retailer is reduced. As it decreases, its profit also decreases, which causes loss for both the supplier and retailer. This model helps determine which type of credit period, whether full or partial, will be profitable for the business and how much length of credit period he/she can provide the retailer to maintain his/her profit without more interest loss. Also, in the case of a partial credit period, if asked to pay how much portion of the purchase at the time of order, the retailer will be interested in buying the item, as a result of which the profit of both will be maintained.

– This model helps the whole system to maximize the profit structure and decides how the decision variables impact one another. Since from sensitivity analysis, when the retailer increases his/her markup, the customers demand decreases automatically due to the products high price. So, the retailer has to reduce his ordering quantity. Low ordering quantity decreases the supplier's profit. But the retailer's own profit and the whole system's profit increase for a certain time. So, this markup is important. Our proposed model helps the retailer how he/she decides the maximum markup for the maximum profit.

8. Conclusion and future scopes

This paper considered an integrated inventory model for a system with a supplier, a retailer and some customers. In this model, we have considered a non-instantaneous deteriorating product which remains fresh for a certain time. Here, we have discussed the credit period where the supplier has offered a credit period to the retailer to increase its profit. In this credit policy, when the supplier offers a full credit period, then the retailer pays his purchasing cost at the end of this period and in a partial credit period, the retailer pays a portion of purchasing cost at delivery time and the remaining pay during allowing time. The retailer also advertises its product to increase the demand of his/her customer. However, in this model, the retailer faces uncertain demand from his/her customer, which increases the chance of shortages. Initially, the retailer first calculates his/her order quantity and then when the supplier offers the credit period, to observe the length of this credit period, the retailer increases his/her ordering quantity depending upon this credit period. In this model, the retailer faces inventory shortages and this shortage is partially backordered. Thus, some customers wait for backorders and some go to other retailers to satisfy their requirements.

This model solves various cases of the length of this credit period. By illustrating two cases of credit periods numerically, it is observed that a full credit period gives better results for the retailer as well as an integrated system profit structure. Though the partial credit gives better results for the supplier. Also, in this model, we explore, what is the maximum number of advertisements that a retailer can place for a certain amount to maximize his/her profit. This study answers how the supplier and the retailer can individually impose the strategy to maximize their profit structure. Also in this model we see how the markup depends on the number of advertisements given by the retailer and also calculate what is the maximum markup that he/she can set for the advertisement.

For future research, the present model can be extended in several ways, such as in this model, one could include the preservation technology for this product deterioration. Another extension of this model could be including the transportation system. Others would generalize this model by including advance payment and discount for this advance payment.

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