

FURTHER RESULTS ON OUTER INDEPENDENT 2-RAINBOW DOMINATING FUNCTIONS OF GRAPHS

BABAK SAMADI* AND NASRIN SOLTANKHAH

Abstract. Let $G = (V(G), E(G))$ be a graph. A function $f : V(G) \rightarrow \mathbb{P}(\{1, 2\})$ is a 2-rainbow dominating function if for every vertex v with $f(v) = \emptyset$, $f(N(v)) = \{1, 2\}$. An outer-independent 2-rainbow dominating function (OI2RD function) of G is a 2-rainbow dominating function f for which the set of all $v \in V(G)$ with $f(v) = \emptyset$ is independent. The outer independent 2-rainbow domination number (OI2RD number) $\gamma_{oir2}(G)$ is the minimum weight of an OI2RD function of G . In this paper, we first prove that $n/2$ is a lower bound on the OI2RD number of a connected claw-free graph of order n and characterize all such graphs for which the equality holds, solving an open problem given in an earlier paper. In addition, a study of this parameter for some graph products is carried out. In particular, we give a closed (resp. an exact) formula for the OI2RD number of rooted (resp. corona) product graphs and prove upper bounds on this parameter for the Cartesian product and direct product of two graphs.

Mathematics Subject Classification. 05C69, 05C76.

Received January 3, 2023. Accepted June 23, 2023.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we consider G as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [14] as a reference for terminology and notation which are not explicitly defined here. The *open neighborhood* of a vertex v is denoted by $N(v)$, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A *leaf* is a vertex of degree one, while its neighbor is a *support vertex*. A *strong support vertex* is a vertex with at least two leaf neighbors. Given the subsets $X, Y \subseteq V(G)$, by $[X, Y]$ we mean the set of all edges with one end point in X and the other in Y .

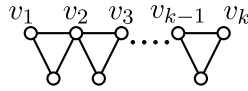
Given a graph G , a subset $X \subseteq V(G)$ is called *independent* if no two distinct vertices of X are adjacent. The size of a largest independent set is called the *independence number* of G and denoted by $\alpha(G)$. A *vertex cover* of G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. The *vertex cover number* $\beta(G)$ is the minimum cardinality among all vertex cover sets of G . A subset $S \subseteq V(G)$ is said to be a *dominating set* (resp. *total dominating set*) in G if every vertex in $V(G) \setminus S$ (resp. $V(G)$) is adjacent to a vertex in S . The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G .

Domination presents a model for a situation in which every empty location (vertex with no guards) needs to be protected by a guard occupying a neighboring location. A generalization of domination was proposed

Keywords. Outer-independent rainbow domination, claw-free graphs, Cartesian product, direct product, rooted product, corona product.

Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran.

*Corresponding author: b.samadi@alzahra.ac.ir; soltan@alzahra.ac.ir

FIGURE 1. A k -unit.

in [1], where different types of guards are deployed, and the empty locations must have all types of guards in their neighborhoods. This relaxation led to the definition of k -rainbow domination. Indeed, a function $f : V(G) \rightarrow \mathbb{P}(\{1, \dots, k\})$ is a k -rainbow dominating function (k RD function) if for every vertex v with $f(v) = \emptyset$, $f(N(v)) = \{1, \dots, k\}$. The k -rainbow domination number $\gamma_{rk}(G)$ is the minimum weight of $\sum_{v \in V(G)} |f(v)|$ taken over all k RD functions of G . This concept was formally defined by Brešar *et al.* [1].

The existence of two adjacent locations with no guards can jeopardize them. Indeed, they would be considered more vulnerable. One improved situation for a location with no guards is to be surrounded by locations in which guards are stationed. This motivates us to consider a k RD function f for which the set of vertices assigned $\emptyset$ under f is independent. More formally, we have the following definition. A function f is an *outer independent k -rainbow dominating function* (OI k RD function) of G if f is a k RD function and the set of vertices with weight $\emptyset$ is an independent set. The *outer independent k -rainbow domination number* (OI k RD number) $\gamma_{oirk}(G)$ is the minimum weight of an OI k RD function of G . An OI k RD function of weight $\gamma_{oirk}(G)$ is called a $\gamma_{oirk}(G)$ -function. This concept was first introduced by Kang *et al.* [8] and studied in [2, 10]. Mansouri and Mojdeh [10] showed that the problem of computing the OI2RD number is NP-hard even when restricted to planar graphs with maximum degree at most four and triangle-free graphs.

In this paper, emphasizing the case $k = 2$, we first provide a characterization of all connected claw-free graphs whose OI2RD numbers are equal to half of their orders, solving an open problem from [10]. In the second section of the paper, we investigate the OI2RD numbers of some graph products such as the Cartesian, direct, rooted and corona products of graphs. We refer the readers to the book [6] for a comprehensive survey of the graph products.

For any function $f : V(G) \rightarrow \mathbb{P}(\{1, 2\})$, we let $V_\emptyset$, $V_{\{1\}}$, $V_{\{2\}}$ and $V_{\{1,2\}}$ stand for the set of vertices assigned with $\emptyset$, $\{1\}$, $\{2\}$ and $\{1, 2\}$ under f , respectively. Since these four sets determine f , we can equivalently write $f = (V_\emptyset, V_{\{1\}}, V_{\{2\}}, V_{\{1,2\}})$. Note that $w(f) = |V_{\{1\}}| + |V_{\{2\}}| + 2|V_{\{1,2\}}|$ is the weight of f .

2. CLAW-FREE GRAPHS

Mansouri and Mojdeh [10] proved that the OI2RD number of a $K_{1,r}$ -free graph G of order n with s' strong support vertices can be bounded from below by $2(n + s')/(1 + r)$. They also posed the open problem of characterizing all $K_{1,r}$ -free (or at least claw-free) graphs for which the lower bound holds with equality. Our aim in this section is to solve the problem for the claw-free graphs (that is, the case $r = 3$). Let G be a claw-free graph with components $G_1, \dots, G_t$. Since $\gamma_{oir2}(G) = \sum_{i=1}^t \gamma_{oir2}(G_i)$, it follows that $\gamma_{oir2}(G) = (n + s')/2$ if and only if $\gamma_{oir2}(G_i) = (|V(G_i)| + s'_i)/2$ for each $1 \leq i \leq t$, in which s'_i is the number of strong support vertices of G_i . In such a case, $s' = \sum_{i=1}^t s'_i = 0$, unless $G_i = P_3$ for some $1 \leq i \leq t$. For such a component, we have $\gamma_{oir2}(G_i) = (|V(G_i)| + s'_i)/2$. So, in what follows, we may assume that G is connected and $s' = 0$. In order to solve the problem in such a case, it suffices to characterize all connected claw-free graphs G of order n for which the equality holds in the lower bound $n/2$ on $\gamma_{oir2}(G)$.

In this section, we show that the OI2RD number of a claw-free graph can be bounded from below by half of its order. In order to characterize all claw-free graphs attaining this bound, we call a graph of the form depicted in Fig. 1 a k -unit in which the number of triangles is $k - 1$. We say v_1 and v_k are the *endpoints* of the k -unit.

Note that a 1-unit is isomorphic to K_1 . We now let $\mathcal{G}$ be the family of all graphs of the form G_1 , G_2 and G_3 depicted in Figure 2. Note that the graphs of G_1 -type can be considered as generalizations of the graphs of G_2 -type when $r = 1$ and the vertex not in the k_1 -unit is adjacent to its both endpoints.

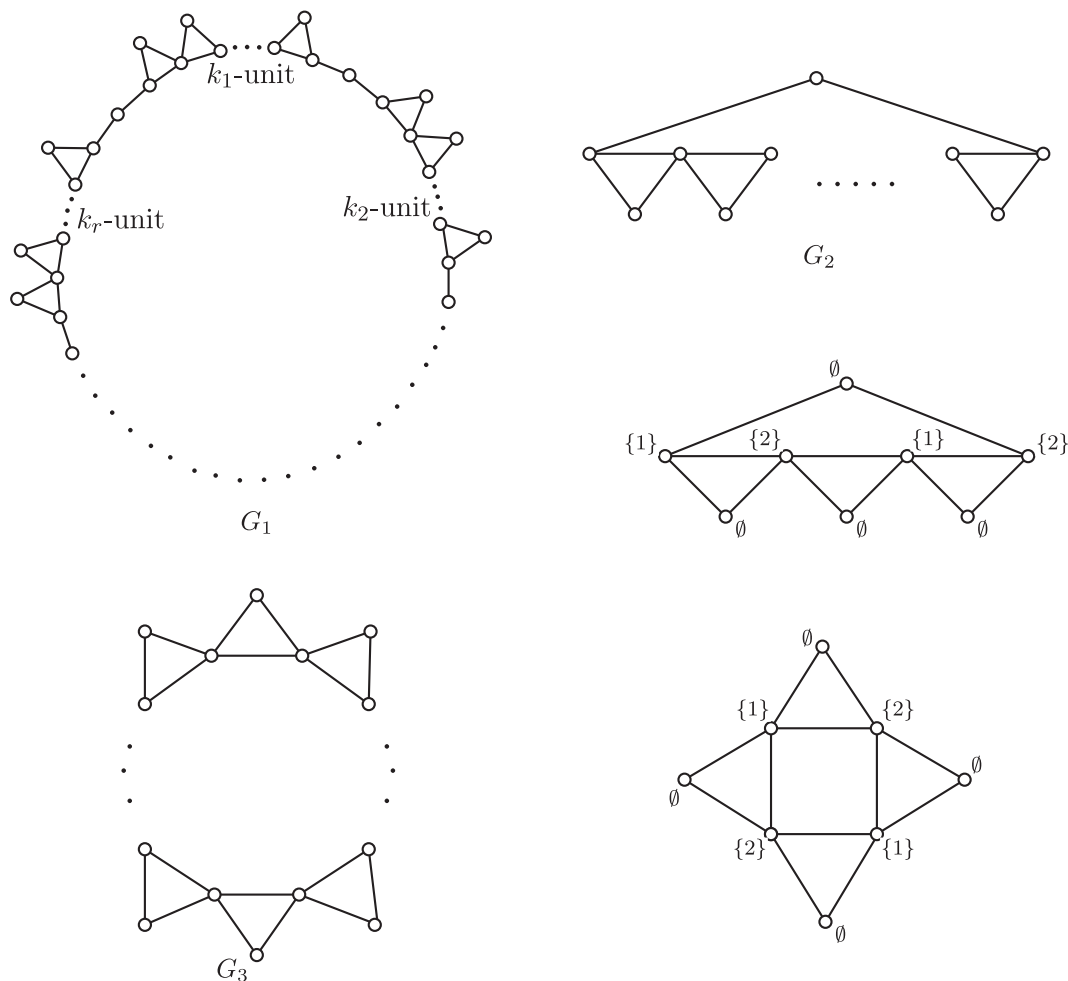


FIGURE 2. The claw-free graphs G_1 , G_2 and G_3 . In G_2 (resp. G_3), the number of triangles is odd (resp. even). In G_1 , $k_1 + \dots + k_r$ is even. Here two simple examples of the graphs of G_i -types, $i \in \{2, 3\}$, with minimum OI2RD functions are given.

Theorem 2.1. *Let G be a connected claw-free graph of order n . Then, $\gamma_{oir2}(G) \geq n/2$ with equality if and only if $G \in \mathcal{G}$.*

Proof. Let f be a $\gamma_{oir2}(G)$ -function. We set $Q = V_\emptyset \cap N(V_{\{1,2\}})$. Since G is a claw-free graph and because $V_\emptyset$ is independent, every vertex in $V_{\{1,2\}}$ has at most two neighbors in Q . Thus, $|Q| \leq 2|V_{\{1,2\}}|$. On the other hand, every vertex in $V_\emptyset \setminus Q$ has at least two neighbors in $V_{\{1\}} \cup V_{\{2\}}$. This implies that $2|V_\emptyset \setminus Q| \leq |[V_\emptyset \setminus Q, V_{\{1\}} \cup V_{\{2\}}]| \leq 2(|V_{\{1\}}| + |V_{\{2\}}|)$. So, $|V_\emptyset \setminus Q| \leq |V_{\{1\}}| + |V_{\{2\}}|$. We now have

$$\begin{aligned} 2(n - \gamma_{oir2}(G)) &\leq 2(n - |V_{\{1\}}| - |V_{\{2\}}| - |V_{\{1,2\}}|) = 2|V_\emptyset| = 2|Q| + 2|V_\emptyset \setminus Q| \\ &\leq 2(|V_{\{1\}}| + |V_{\{2\}}| + 2|V_{\{1,2\}}|) = 2\gamma_{oir2}(G), \end{aligned} \quad (1)$$

implying the lower bound.

Suppose that the equality holds for a connected claw-free graph G . Then, all inequalities in (1) necessarily hold with equality. In particular, $V_{\{1,2\}} = \emptyset$ (and consequently $Q = \emptyset$) by the equality instead of the first

inequality in (1). This implies that every vertex in $V_\emptyset$ has at least one neighbor in each of $V_{\{1\}}$ and $V_{\{2\}}$. Taking this fact into account, the resulting equality $2|V_\emptyset| = |[V_\emptyset, V_{\{1\}} \cup V_{\{2\}}]|$ shows that every vertex in $V_\emptyset$ has precisely one neighbor in each of $V_{\{1\}}$ and $V_{\{2\}}$. On the other hand, $|[V_\emptyset, V_{\{1\}} \cup V_{\{2\}}]| = 2(|V_{\{1\}}| + |V_{\{2\}}|)$ follows that every vertex in $V_{\{1\}} \cup V_{\{2\}}$ is adjacent to exactly two vertices in $V_\emptyset$.

Let $H = G[V_{\{1\}} \cup V_{\{2\}}]$. Suppose to the contrary that $\deg_H(v) \geq 3$ for some $v \in V(H)$. Since G is claw-free and because v has two neighbors in $V_\emptyset$, it follows that every vertex in $N_H(v)$ must be adjacent to at least one of the two neighbors of v , say x_1 and x_2 , in $V_\emptyset$. This implies that $\deg(x_1) \geq 3$ or $\deg(x_2) \geq 3$, a contradiction. The above discussion guarantees that $\Delta(H) \leq 2$, and thus H is isomorphic to a disjoint union of some cycles and paths.

Let H' be a cycle $v_1v_2 \cdots v_tv_1$ as a component of H . Let v_{11} and v_{12} be the neighbors of v_1 in $V_\emptyset$. Since G is claw-free, both v_2 and v_t have at least one neighbor in $\{v_{11}, v_{12}\}$. Furthermore, because $\deg(v_{11}) = \deg(v_{12}) = 2$, both v_2 and v_t have exactly one neighbor in $\{v_{11}, v_{12}\}$. We may assume, without loss of generality, that $v_{11}v_t, v_{12}v_2 \in E(G)$. Let v_{22} be the second neighbor of v_2 in $V_\emptyset$. Again, because G is claw-free and $v_{12}v_{22}, v_{12}v_3 \notin E(G)$, we infer that $v_{22}v_3 \in E(G)$. Iterating this process results in a graph of the form G_3 , in which $v_1v_2 \cdots v_tv_1$ is the resulting cycle by removing all vertices of degree two. In addition, since all the vertices in $V_\emptyset$ have degree two, both $\{1\}$ and $\{2\}$ must appear on their neighbors on the cycle. This implies that t is even. Note that each vertex of H' has no other neighbors in G . Now G is of the form G_3 by its connectedness.

In what follows, we may assume that H does not have any cycle as a component. If H is an edgeless graph, then G is isomorphic to the cycle C_n . Notice that

$$\gamma_{oir2}(C_p) = \lfloor p/2 \rfloor + \lceil p/4 \rceil - \lfloor p/4 \rfloor \quad (2)$$

for $p \geq 3$ (see [10]). Since $\gamma_{oir2}(C_n) = n/2$, the formula (2) shows that $n \equiv 0 \pmod{4}$. It is then easy to observe that G is of the form of G_1 , in which $r = n/2$ and $k_1 = \cdots = k_r = 1$. Suppose now that H is not edgeless and let H'' be a path $v_1v_2 \cdots v_t$, on $t \geq 2$ vertices, as a component of H . Let v_2 be adjacent to two vertices v_{21} and v_{22} in $V_\emptyset$. Note that v_1 must be adjacent to at least one of v_{21} and v_{22} , for otherwise G would have a claw as an induced subgraph. If v_1 is adjacent to both v_{21} and v_{22} , then $G[v_1, v_2, v_{21}]$ is a 2-unit and $G \cong G[v_1, v_2, v_{21}, v_{22}] \cong K_4 - v_{21}v_{22}$. In such a case, G is of the form G_2 since it is connected. So, $G \in \mathcal{G}$. In what follows, we assume that v_1 is adjacent to only one of v_{21} and v_{22} . We then proceed with v_2 . Since G is claw-free, it follows that both v_1 and v_3 must have neighbors in $\{v_{21}, v_{22}\}$. Moreover, $\deg(v_{21}) = \deg(v_{22}) = 2$ shows that both v_1 and v_3 have exactly one neighbor in $\{v_{21}, v_{22}\}$. So, we may assume that $v_1v_{21}, v_3v_{22} \in E(G)$. Similarly, v_3 has two neighbors v_{31} and v_{32} in $V_\emptyset$, in which we may assume that $v_{31} = v_{22}$. By repeating this process we obtain a t -unit on the set of vertices $K = \{v_1, v_2, \dots, v_t, u_{12}, u_{23}, \dots, u_{(t-1)t}\}$ in which $u_{i(i+1)}$ is adjacent to both v_i and v_{i+1} , for $1 \leq i \leq t-1$. We now consider two cases depending on $A = (N(v_1) \cap N(v_t)) \setminus \{v_2\}$.

Case 1. $A \neq \emptyset$. Let u_{1t} be in A . Because the path H'' is a component of H , $A \cap V(H) = \emptyset$. Therefore, $u_{1t} \in V_\emptyset$.

Since both v_1 and v_t have exactly two neighbors in $V_\emptyset$, it follows that $A = \{u_{1t}\}$. In such a situation, the subgraph induced by $K \cup \{u_{1t}\}$ is isomorphic to G because it is connected. Notice that since all vertices in $V_\emptyset$ have degree two, both $\{1\}$ and $\{2\}$ must appear on their neighbors in $\{v_1, \dots, v_t\}$. This shows that t is even and hence G is of the form G_2 .

Case 2. $A = \emptyset$. This implies that the subgraph induced by K is a t -unit. Then v_t is adjacent to a vertex $x' \in V_\emptyset$ and x' is adjacent to vertex $x \notin V(H) \setminus V(H'')$ (note that if $x \in V(H'')$, then $x = v_j \in \{v_1, \dots, v_{t-1}\}$. If $j = 1$, then G is of the form G_2 and hence $G \in \mathcal{G}$. If $j \geq 2$, then v_j has at least three neighbors in $V_\emptyset$ which is impossible). Let x belong to a component H''' of H . Since H does not have a cycle as a component, it follows that H''' is a path. In such a case, the vertices of H''' belong to a $|V(H''')|$ -unit by a similar fashion. Iterating this process we obtain some $|V(H_1)|, \dots, |V(H_s)|$ -units constructed as above, in which $H_1 = H''$ and s is the largest integer for which there exists such a $|V(H_s''')|$ -unit. Let $H_s = w_1 \cdots w_p$ and w_p be the vertex which has only one neighbor in $V_\emptyset$ in the subgraph induced by $V(H_1) \cup \dots \cup V(H_r)$. Similar to Case 1, there exists a vertex $u_{1p} \in V_\emptyset$ adjacent to both v_1 and w_p . On the other hand, both $\{1\}$ and $\{2\}$ must appear on the neighbors of each vertex in $V_\emptyset$. This implies that $\sum_{i=1}^r |V(H_i)|$ must be even. Therefore, G is of the form G_1 .

In both cases above, we have concluded that $G \in \mathcal{G}$.

Conversely, let $G \in \mathcal{G}$. Suppose first that G is of the form G_2 . Let $v_1 \cdots v_{2t}$ be the path on the set of vertices of degree at least three of G_2 . Then $(f(v_{2i-1}), f(v_{2i})) = (\{1\}, \{2\})$ for $1 \leq i \leq t$, and $f(v) = \emptyset$ for the other vertices defines an OI2RD function with weight half of the order. Let G be of the form G_1 . Let $\{u_1, \dots, u_{2p}\}$ be the set of vertices of degree at least three of G_1 such that $u_1 \cdots u_{k_1}$ is the path in the k_1 -unit, $u_{k_1+1} \cdots u_{k_1+k_2}$ is the path in the k_2 -unit, and so on. It is easy to see that $(g(u_{2i-1}), g(u_{2i})) = (\{1\}, \{2\})$ for $1 \leq i \leq p$, and $g(u) = \emptyset$ for the other vertices is an OI2RD function with weight $n/2$. Finally, we suppose that G is of the form G_3 . Let $x_1 x_2 \cdots x_{2q} x_1$ be the cycle on the vertices of degree four. Then the assignment $(g(x_{2i-1}), g(x_{2i})) = (\{1\}, \{2\})$ for $1 \leq i \leq q$, and $g(x) = \emptyset$ for the other vertices defines an OI2RD function of G_3 with weight half of its order. Therefore, in all three possibilities, we have concluded that $\gamma_{oir2}(G) = n/2$. This completes the proof. $\square$

3. OI2RD NUMBER OF SOME GRAPH PRODUCTS

The concept of domination in graph products has been extensively studied. In fact, many authors have focused on bounding $\eta(G * H)$ from lower and above in terms of well-known graph parameters, where η is a domination parameter and $*$ is a graph product. For more information on these topics, the reader may consult [6, 7]. For instance, Vizing [13] in 1968 posed the following conjecture for any Cartesian product graph $G \square H$, which is still open:

$$\gamma(G \square H) \geq \gamma(G)\gamma(H).$$

Regarding rainbow domination, Pilipczuk *et al.* [11] proved that

$$\gamma_{rk}(G \square H) \geq \frac{k}{k+1} \gamma(G)\gamma(H)$$

for all graphs G and H and every $k \geq 1$. Moreover, it was proved in [9] that $\gamma_{r2}(G \circ H) = 2\gamma_t(G)$ for any non-trivial connected graphs G and H with $\gamma_{r2}(H) \geq 4$ (recall that in the *lexicographic product* graph $G \circ H$ the vertex set is $V(G) \times V(H)$, and two distinct vertices (g, h) and (g', h') are adjacent if either “ $gg' \in E(G)$ ” or “ $g = g'$ and $hh' \in E(H)$ ”).

Here we consider the OI2RD number of direct, Cartesian, rooted and corona products of two graphs (note that in the lexicographic case, an exact formula was given in [2]).

3.1. Direct and Cartesian products

For the following two standard products of graphs G and H (see [6]), the vertex set of the product is $V(G) \times V(H)$. In the edge set of the *direct product* $G \times H$, two vertices are adjacent if they are adjacent in both coordinates. On the other hand, in the edge set of the *Cartesian product* $G \square H$, two vertices are adjacent if they are adjacent in one coordinate and equal in the other.

For a graph G , we let I_G denote the set of isolated vertices of G . By G^- we denote the graph obtained from G by removing all the isolated vertices of G . We observe that $I = (I_G \times V(H)) \cup (V(G) \times I_H)$ is the set of isolated vertices of the direct product $G \times H$ with $|I| = |I_G||V(H)| + |I_H||V(G)| - |I_G||I_H|$. On the other hand, $\gamma_{oir2}(G \times H) = \gamma_{oir2}(G^- \times H^-) + |I|$. So, we may suppose that both G and H have no isolated vertices.

Theorem 3.1. *Let G and H be two graphs with no isolated vertices. Then,*

$$\gamma_{oir2}(G \times H) \leq \min\{\gamma_{oir2}(H)|V(G)|, \gamma_{oir2}(G)|V(H)|\}.$$

Moreover, this bound is sharp.

Proof. Let g be a $\gamma_{oir2}(H)$ -function. We define $f : V(G) \times V(H) \rightarrow \mathbb{P}(\{1, 2\})$ by $f(x, y) = g(y)$ for each $(x, y) \in V(G) \times V(H)$. Suppose that $(x, y)(x', y') \in E(G \times H)$ for some $(x, y), (x', y') \in V_\emptyset^f$. This shows that $yy' \in E(H)$ and that $g(y) = g(y') = \emptyset$. This is a contradiction. Therefore, $V_\emptyset^f$ is an independent set in $G \times H$.

For any $(x, y) \in V(G) \times V(H)$ with weight $\emptyset$ under f , the equality $g(y) = \emptyset$ implies that $g(N_H(y)) = \{1, 2\}$. Moreover, $N_G(x) \neq \emptyset$ since G has no isolated vertex. So, $f(N_{G \times H}(x, y)) = f(N_G(x) \times N_H(y)) = g(N_H(y)) = \{1, 2\}$. Therefore, f is an OI2RD function of $G \times H$. We then have $\gamma_{oir2}(G \times H) \leq \omega(f) = \gamma_{oir2}(H)|V(G)|$. Interchanging the roles of G and H establishes the upper bound.

That the bound is sharp, can be seen as follows. We consider the graph $K_m \times K_n$ with $m, n \geq 2$. By the structure, at most $\alpha(K_m \times K_n) = \max\{m, n\}$ vertices of $K_m \times K_n$ are assigned $\emptyset$ under any $\gamma_{oir2}(G \times H)$ -function. This shows that

$$\gamma_{oir2}(K_m \times K_n) \geq mn - \max\{m, n\} = \min\{m(n-1), n(m-1)\},$$

satisfying the equality in the upper bound. $\square$

Regarding the Cartesian product $G \square H$, we observe that

$$\gamma_{oir2}(G \square H) = \gamma_{oir2}(G^- \square H^-) + |I_G| \gamma_{oir2}(H) + |I_H| \gamma_{oir2}(G) - |I_G| |I_H|.$$

With this in mind, it suffices to assume that both G and H have no isolated vertices.

Theorem 3.2. *Let G and H be two graphs with no isolated vertices. Then,*

$$\gamma_{oir2}(G \square H) \leq \alpha(G)\beta(H) + \beta(G)|V(H)| - \min\{\beta(G), \beta(H)\}.$$

Furthermore, this bound is sharp.

Proof. Without loss of generality, we may assume that $\beta(G) \leq \beta(H)$. Let I and J denote an $\alpha(G)$ -set and an $\alpha(H)$ -set, respectively. Suppose that $V(G) \setminus I = \{g_1, \dots, g_r\}$ and $V(H) \setminus J = \{h_1, \dots, h_s\}$, where $r = \beta(G) \leq \beta(H) = s$ by our assumption and the well-known Gallai theorem [3] (which states that $\alpha(F) + \beta(F) = |V(F)|$ for any graph F). It is a routine matter to see that

$$S = \{(g_i, h_i) \mid 1 \leq i \leq r\} \cup (I \times J)$$

is independent in $G \square H$. Note that the subsets

$$S, \quad I \times (V(H) \setminus J) \quad \text{and} \quad ((V(G) \setminus I) \times V(H)) \setminus \{(g_i, h_i) \mid 1 \leq i \leq r\}$$

form a partition of $V(G) \times V(H)$. We now define $h : V(G) \times V(H) \rightarrow \mathbb{P}(\{1, 2\})$ by

$$h((x, y)) = \begin{cases} \emptyset & \text{if } (x, y) \in S, \\ \{1\} & \text{if } (x, y) \in I \times (V(H) \setminus J), \\ \{2\} & \text{if } (x, y) \in ((V(G) \setminus I) \times V(H)) \setminus \{(g_i, h_i) \mid 1 \leq i \leq r\}. \end{cases}$$

Notice that $S = V_\emptyset^h$ is independent in $G \square H$ as mentioned above. Now let $(x, y) \in S$. We distinguish two possibilities depending on membership of (x, y) .

- (i) Suppose that $(x, y) \in I \times J$. Since H has no isolated vertices, y is adjacent to a vertex $y' \in V(H) \setminus J$. So, (x, y) is adjacent to (x, y') with $h((x, y')) = \{1\}$. Similarly, since G has no isolated vertices, x is adjacent to a vertex $x' \in V(G) \setminus I$. Moreover, $y \notin \{h_1, \dots, h_s\}$. This shows that (x, y) is adjacent to (x', y) with $h((x', y)) = \{2\}$. Therefore, $h(N_{G \square H}((x, y))) = \{1, 2\}$.
- (ii) Suppose that $(x, y) = (g_i, h_i)$ for some $1 \leq i \leq r$. Since I is an $\alpha(G)$ -set, x has a neighbor $x' \in I$. Therefore, (x, y) is adjacent to $(x', y) \in I \times (V(H) \setminus J)$ for which $h((x', y)) = \{1\}$. Because H has no isolated vertices, there exists a vertex $y' \in V(H)$ adjacent to y . So, (x, y) is adjacent to (x, y') . This implies that $(x, y') \notin \{(g_i, h_i) \mid 1 \leq i \leq r\}$ as this set is independent. This shows that, (x, y) is adjacent to $(x, y') \in ((V(G) \setminus I) \times V(H)) \setminus \{(g_i, h_i) \mid 1 \leq i \leq r\}$ with $h((x, y')) = \{2\}$. Therefore, $h(N_{G \square H}((x, y))) = \{1, 2\}$.

The above discussion guarantees that h is an OI2RD function of $G \square H$. Thus,

$$\gamma_{oir2}(G \square H) \leq \omega(h) = \alpha(G)(|V(H)| - \alpha(H)) + (|V(G)| - \alpha(G))|V(H)| - \beta(G). \quad (3)$$

On the other hand, we have $\alpha(F) + \beta(F) = |V(F)|$ for any graph F . Taking this fact into account, the desired upper bound follows from (3).

That the upper bound is sharp, may be seen by considering $G = P_m$ and $H = K_n$ for $m, n \geq 2$ and $n \geq \lfloor m/2 \rfloor + 1$. It is easily observed that $\gamma_{oir2}(P_m \square K_n) = m(n-1)$. Moreover, $m(n-1) = \alpha(P_m)\beta(K_n) + \beta(P_m)|V(K_n)| - \beta(P_m)$. This completes the proof. $\square$

3.2. Rooted and corona products

A *rooted graph* is a graph in which one vertex is labeled in a special way to distinguish it from the other vertices. The special vertex is called the *root* of the graph. Let G be a labeled graph on n vertices. Let $\mathcal{H}$ be a sequence of n rooted graphs $H_1, \dots, H_n$. The *rooted product graph* $G(\mathcal{H})$ is the graph obtained by identifying the root of H_i with the i th vertex of G (see [5]). We here consider the particular case of rooted product graphs where $\mathcal{H}$ consists of n isomorphic rooted graphs [12]. More formally, assuming that $V(G) = \{g_1, \dots, g_n\}$ and that the root vertex of H is v , we define the rooted product graph $G \circ_v H = (V, E)$, where $V = V(G) \times V(H)$ and

$$E = \bigcup_{i=1}^n \{(g_i, h)(g_i, h') \mid hh' \in E(H)\} \cup \{(g_i, v)(g_j, v) \mid g_i g_j \in E(G)\}.$$

Note that subgraphs induced by H -layers of $G \circ_v H$ are isomorphic to H . We next study the OI2RD number of rooted product graphs.

Theorem 3.3. *Let G be any graph of order n . If H is any graph with root v , then*

$$\gamma_{oir2}(G \circ_v H) \in \{n\gamma_{oir2}(H) - \alpha(G), n\gamma_{oir2}(H), n\gamma_{oir2}(H) + \beta(G)\}.$$

Proof. We first prove that

$$n\gamma_{oir2}(H) - \alpha(G) \leq \gamma_{oir2}(G \circ_v H) \leq n\gamma_{oir2}(H) + \beta(G). \quad (4)$$

In order to prove the lower bound, let f be a $\gamma_{oir2}(G \circ_v H)$ -function. If $f_x = f|_{(G \circ_v H)[\{x\} \times V(H)]}$ is an OI2RD function of $H_x = (G \circ_v H)[\{x\} \times V(H)] \cong H$ for every $x \in V(G)$, then $\gamma_{oir2}(H) \leq \omega(f_x)$ for all $x \in V(G)$. Therefore,

$$\gamma_{oir2}(G \circ_v H) = \omega(f) = \sum_{x \in V(G)} \omega(f_x) \geq n\gamma_{oir2}(H) > n\gamma_{oir2}(H) - \alpha(G).$$

So, in what follows, we may assume that f_x is not an OI2RD function of H_x for some $x \in V(G)$. Since f is an OI2RD function of $G \circ_v H$, there are no two adjacent vertices of H_x which are assigned $\emptyset$ under f_x . Therefore, there exists a vertex (x, y) of H_x with $f_x((x, y)) = \emptyset$ for which $f_x(N_{H_x}((x, y))) \neq \{1, 2\}$. By the structure of rooted products and since f is an OI2RD function of $G \circ_v H$, it follows that $y = v$. We now consider two cases.

Case 1. Let (x, v) be an isolated vertex of H_x . In such a situation, $G \circ_v H$ is isomorphic to the disjoint union of one copy of G and n copies of the graph $H - v$. On the other hand,

$$\begin{aligned} \omega(f|_{V(G)}) &= \gamma_{oir2}(G) = |V_{\{1\}} \cap V(G)| + |V_{\{2\}} \cap V(G)| + 2|V_{\{1,2\}} \cap V(G)| \\ &\geq n - |V_{\emptyset} \cap V(G)| \geq n - \alpha(G) = \beta(G). \end{aligned}$$

Therefore, $\gamma_{oir2}(G \circ_v H) = \gamma_{oir2}(G) + n\gamma_{oir2}(H - v) = \gamma_{oir2}(G) + n(\gamma_{oir2}(H) - 1) \geq \beta(G) + n(\gamma_{oir2}(H) - 1) = n\gamma_{oir2}(H) - \alpha(G)$.

Case 2. Suppose that (x, v) is not an isolated vertex of H_x . Since $f_x((x, v)) = \emptyset$, it follows that (x, v) is adjacent to a vertex (x, w) of H_x for which $f_x((x, w)) = \{1\}$ or $\{2\}$. Now the assignment $f'_x((x, v)) = \{1\}$ and $f'_x((x, u)) = f_x((x, u))$ for the other vertices $u \in V(H)$ defines an OI2RD function of H_x with weight $\omega(f_x) + 1$. Therefore, $\gamma_{oir2}(H) \leq \omega(f_x) + 1$ for each vertex $x \in V(G)$ for which f_x is not an OI2RD function of H_x .

Now let S be a $\beta(G)$ -set. This shows that at least $\beta(G)$ vertices in $V(G) \times \{v\}$ are assigned at least $\{1\}$ or $\{2\}$ under f . Moreover, f_x is an OI2RD function of H_x for each vertex x with $f((x, v)) \neq \emptyset$. Therefore,

$$\begin{aligned} \gamma_{oir2}(G \circ_v H) &= \sum_{x \in V(G)} \omega(f_x) = \sum_{x \in V(G) \setminus S} \omega(f_x) + \sum_{x \in S} \omega(f_x) \\ &\geq (n - \beta(G))(\gamma_{oir2}(H) - 1) + \beta(G)\gamma_{oir2}(H) = n\gamma_{oir2}(H) - \alpha(G). \end{aligned}$$

We now prove the upper bound. Suppose that there exists a $\gamma_{oir2}(H)$ -function f for which $f(v) \neq \emptyset$. Clearly, f results in a $\gamma_{oir2}(H_x)$ -function f_x for each $x \in V(G)$. Therefore, $\gamma_{oir2}(G \circ_v H) \leq \sum_{x \in V(G)} \omega(f_x) = n\gamma_{oir2}(H) \leq n\gamma_{oir2}(H) + \beta(G)$. Assume now that every $\gamma_{oir2}(H)$ -function f assigns $\emptyset$ to v . Since the vertices with weight $\emptyset$ under f are independent, it follows that one 1 or one 2, say one 2, belongs to $f(N_H(v))$. Now let S be a $\beta(G)$ -set. We define $g : V(G) \times V(H) \rightarrow \mathbb{P}(\{1, 2\})$ by

$$g((x, y)) = \begin{cases} f(y) & \text{if } y \neq v, \\ \{1\} & \text{if } y = v \text{ and } x \in S, \\ \emptyset & \text{if } y = v \text{ and } x \in V(G) \setminus S. \end{cases}$$

It is then easy to check that g is an OI2RD function of $G \circ_v H$ with weight $\omega(g) = n\gamma_{oir2}(H) + \beta(G)$, implying the upper bound.

Note that if G is edgeless, then $\gamma_{oir2}(G \circ_v H) = n\gamma_{oir2}(H) = n\gamma_{oir2}(H) + \beta(G)$. So, in what follows we assume that G is not edgeless. We distinguish the following cases depending on the behavior of $\gamma_{oir2}(H)$ -functions.

Case 3. Let every $\gamma_{oir2}(H)$ -function assign $\emptyset$ to v . Let f be a $\gamma_{oir2}(G \circ_v H)$ -function. Suppose to the contrary that there exists a vertex $x \in V(G)$ for which $\omega(f_x) \leq \gamma_{oir2}(H) - 1$, where $f_x = f|_{\{x\} \times V(H)}$. By the properties of the rooted product graph $G \circ_v H$ and since f_x is not an OI2RD function of H_x , we have $f(x, v) = f_x((x, v)) = \emptyset$ and $|f(N_{H_x}(x, v))| = 1$. Now $g(y) = f_x(x, y)$ for $y \in V(H) \setminus \{v\}$, and $g(v) = \{1\}$ defines an OI2RD function of H with $\omega(g) = \gamma_{oir2}(H)$ for which $g(v) \neq \emptyset$. This is a contradiction. Therefore, $\omega(f_x) \geq \gamma_{oir2}(H)$ for each $x \in V(G)$. Set $A = \{x \in V(G) \mid \omega(f_x) = \gamma_{oir2}(H)\}$ and $B = \{x \in V(G) \mid \omega(f_x) > \gamma_{oir2}(H)\}$. Obviously, $|A| + |B| = n$. We then have

$$\omega(f) \geq |A|\gamma_{oir2}(H) + |B|(\gamma_{oir2}(H) + 1) = n\gamma_{oir2}(H) + |B|. \quad (5)$$

If $f_x((x, v)) \neq \emptyset$ for some vertex $x \in A$, then it is easy to see that $g(y) = f_x((x, y))$ for all $y \in V(G)$ is an OI2RD function of H with weight $\gamma_{oir2}(H)$ for which $g(v) \neq \emptyset$. This is a contradiction. Therefore, $f_x((x, v)) = \emptyset$ for all $x \in A$. This implies that $\{(x, v) \mid x \in A\}$ is independent in $(G \circ_v H)[V(G) \times \{v\}] \cong G$. Hence $|A| \leq \alpha(G)$, implying that $|B| = n - |A| \geq \beta(G)$. This results in $\gamma_{oir2}(G \circ_v H) = \omega(f) \geq n\gamma_{oir2}(H) + \beta(G)$ by (5). Therefore, $\gamma_{oir2}(G \circ_v H) = n\gamma_{oir2}(H) + \beta(G)$ by (4).

Case 4. Suppose that $g(v) \neq \emptyset$ for some $\gamma_{oir2}(H)$ -function g . We need to consider two subcases depending on the collection of such functions g .

Subcase 4.1. Let there exist such a function g under which v is not adjacent to any vertex of H with weight $\emptyset$. Notice that at least one 1 or one 2, say one 1, belongs to $g(N_H(v))$ in H . Let I be an $\alpha(G)$ -set. Then, $h : V(G) \times V(H) \rightarrow \mathbb{P}(\{1, 2\})$ defined by

$$h((x, y)) = \begin{cases} g(y) & \text{if } x \in V(G) \text{ and } y \neq v, \\ \emptyset & \text{if } x \in I \text{ and } y = v, \\ \{2\} & \text{if } x \in V(G) \setminus I \text{ and } y = v, \end{cases}$$

is an OI2RD function of $G \circ_v H$ with weight $n\gamma_{oir2}(H) - \alpha(G)$. This implies that $\gamma_{oir2}(G \circ_v H) = n\gamma_{oir2}(H) - \alpha(G)$ in view of (4).

Subcase 4.2. Suppose now that for all such functions g , v is adjacent to a vertex of H with weight $\emptyset$ under g . Note that $h((x, y)) = g(y)$ for all $x \in V(G)$ and $y \in V(H)$ defines an OI2RD function of $G \circ_v H$ with weight $n\gamma_{oir2}(H)$. So, $\gamma_{oir2}(G \circ_v H) \leq n\gamma_{oir2}(H)$. We again suppose that f is a $\gamma_{oir2}(G \circ_v H)$ -function. Let $\omega(f_x) \leq \gamma_{oir2}(H) - 1$ for some $x \in V(G)$. This shows that $f_x((x, v)) = \emptyset$, for otherwise f_x would be an OI2RD function of $H_x \cong H$ with a weight less than $\gamma_{oir2}(H)$, which is impossible. Therefore, all neighbors of (x, v) in $\{x\} \times V(H)$ must have nonempty weights under f . But the assignment $h(v) = \{1\}$, and $h(y) = f((x, y))$ for other vertices defines an OI2RD function of H with weight $\gamma_{oir2}(H)$ (that is, a $\gamma_{oir2}(H)$ -function) for which no vertices adjacent to v are assigned $\emptyset$ under h . This contradicts the fact that all $\gamma_{oir2}(H)$ -functions assigning a nonempty weight to v assign $\emptyset$ to a neighbor of it. The above argument guarantees that $\omega(f_x) \geq \gamma_{oir2}(H)$ for all $x \in V(G)$. Therefore, $\gamma_{oir2}(G \circ_v H) = \omega(f) \geq n\gamma_{oir2}(H)$. This results in $\gamma_{oir2}(G \circ_v H) = n\gamma_{oir2}(H)$.

All in all, we have shown that $\gamma_{oir2}(G \circ_v H)$ belongs to $\{n\gamma_{oir2}(H) - \alpha(G), n\gamma_{oir2}(H), n\gamma_{oir2}(H) + \beta(G)\}$. This completes the proof. $\square$

Let G and H be graphs where $V(G) = \{v_1, \dots, v_n\}$. We recall that the corona $G \odot H$ of graphs G and H is obtained from the disjoint union of G and n disjoint copies of H , say $H_1, \dots, H_n$, such that for all $i \in \{1, \dots, n\}$, the vertex $v_i \in V(G)$ is adjacent to every vertex of H_i .

Unlike the cases of Cartesian and direct products, the existence of isolated vertices in H is irrelevant to the number of components of $G \odot H$. In particular, if H has isolated vertices, $G \odot H$ remains connected when G is connected. In fact, as we next show, the exact formula for $\gamma_{oir2}(G \odot H)$ changes in the case when H has isolated vertices. In particular, when $|V(H)| = 1$, it establishes the NP-hardness of the problem of computing γ_{oir2} even for some special families of graphs (see [4, 10]).

Cabrera Martínez [2] proved that $\gamma_{oir2}(G \odot H) = |V(G)|(|V(H)| + 1) - |V(G)|\alpha(H)$ for all graphs G and H with no isolated vertices. In what follows, we present an exact formula for $\gamma_{oir2}(G \odot H)$ for any graph G with no isolated vertices and arbitrary graph H . By the way, the method by which we prove the following theorem is different from that of [2].

Theorem 3.4. *Let G be a graph of order n with no isolated vertices and let H be any graph with i_H isolated vertices. If $|V(H)| = 1$, then*

$$\gamma_{oir2}(G \odot H) = n + \beta(G).$$

If $|V(H)| \geq 2$, then

$$\gamma_{oir2}(G \odot H) = \begin{cases} n(\beta(H) + 1) & \text{if } i_H = 0, \\ n(\beta(H) + 2) & \text{if } i_H \neq 0. \end{cases}$$

Proof. We first suppose that $H \cong K_1$. Let $V(G) = \{v_1, \dots, v_n\}$. Then, $G \odot K_1$ is obtained from G by joining n new vertices $u_1, \dots, u_n$ to $v_1, \dots, v_n$, respectively. Let f be a $\gamma_{oir2}(G \odot K_1)$ -function. Clearly, $1 \leq |f(v_i)| + |f(u_i)| \leq 2$ for each $1 \leq i \leq n$. If $|f(v_i)| + |f(u_i)| = 2$ for some $1 \leq i \leq n$, we may assume that $f(v_i) = \{1, 2\}$ and $f(u_i) = \emptyset$. Moreover, $f(u_i) = \{1\}$ or $\{2\}$ whenever $|f(v_i)| + |f(u_i)| = 1$. Let $A = \{1 \leq i \leq n \mid |f(v_i)| + |f(u_i)| = 1\}$. Note that

$$\gamma_{oir2}(G \odot K_1) = \sum_{i \notin A} (|f(v_i)| + |f(u_i)|) + \sum_{i \in A} (|f(v_i)| + |f(u_i)|) = 2n - |A|. \quad (6)$$

On the other hand, $|A| \leq \alpha(G)$ as the vertices v_i , for which $i \in A$, are assigned $\emptyset$ under f . So, $\gamma_{oir2}(G \odot K_1) \geq 2n - \alpha(G) = n + \beta(G)$ by (6).

Let I be an $\alpha(G)$ -set. We can observe that the assignment $\emptyset$ to the vertices in I , $\{1\}$ to the other vertices of G , and $\{2\}$ to $u_1, \dots, u_n$ defines an OI2RD function of $G \odot K_1$ with weight $n + \beta(G)$. Therefore, $\gamma_{oir2}(G \odot K_1) \leq n + \beta(G)$. This results in the equality for the case when $|V(H)| = 1$.

Now let $|V(H)| \geq 2$. Consider a function $f_G = (V_\emptyset, V_{\{1\}}, V_{\{2\}}, V_{\{1,2\}})$ of G such that $V_\emptyset$ is an independent set. We next define a function $f : V(G \odot H) \rightarrow \mathbb{P}(\{1, 2\})$ as follows. Suppose that $v_i \in V(G) = \{v_1, \dots, v_n\}$.

- (a) If $f_G(v_i) = \emptyset$, we consider a function f_H which assigns $\{1\}$ and $\{2\}$ to the vertices of H_i for which both $\{1\}$ and $\{2\}$ appear at least one time to the vertices of H_i , and let $f(w) = f_H(w)$ for all $w \in V(H_i)$.
- (b) Let $f_G(v_i) = \{1\}$ (resp. $f_G(v_i) = \{2\}$). Suppose that I is an $\alpha(H)$ -set and I_H is the set of isolated vertices of H . Clearly, $I_H \subseteq I$. We define g_H by $g_H(w) = \emptyset$ for each $w \in I \setminus I_H$, and $g_H(w) = \{2\}$ (resp. $g_H(w) = \{1\}$) for the other vertices w of H . We next let $f(w) = g_H(w)$ for each $w \in V(H_i)$.
- (c) Let $f_G(v_i) = \{1, 2\}$. Define k_H by $k_H(w) = \emptyset$ for each $w \in I$, and $k_H(w) = \{1\}$ for the other vertices. We next let $f(w) = k_H(w)$ for each $w \in V(H_i)$.
- (d) For each $1 \leq i \leq n$, let $f(v_i) = f_G(v_i)$.

It is not hard to see that the above mentioned function f is an OI2RD function of $G \odot H$ with weight

$$\begin{aligned} \omega(f) = & |V_\emptyset||V(H)| + |V_{\{1\}}|(|V(H)| - \alpha(H) + i_H + 1) + |V_{\{2\}}|(|V(H)| - \alpha(H) + i_H + 1) \\ & + |V_{\{1,2\}}|(|V(H)| - \alpha(H) + 2). \end{aligned}$$

Since $f_G = (V_\emptyset, V_{\{1\}}, V_{\{2\}}, V_{\{1,2\}})$ is an arbitrary function of G for which $V_\emptyset$ is independent and because $\alpha(H) + \beta(H) = |V(H)|$, we deduce that

$$\gamma_{oir2}(G \odot H) \leq \min\{|V_\emptyset||V(H)| + |V_{\{1\}}|(\beta(H) + i_H + 1) + |V_{\{2\}}|(\beta(H) + i_H + 1) + |V_{\{1,2\}}|(\beta(H) + 2)\},$$

taken over all possible function $f_G = (V_\emptyset, V_{\{1\}}, V_{\{2\}}, V_{\{1,2\}})$ of G for which $V_\emptyset$ is an independent set in G .

On the other hand, let $g = (V'_\emptyset, V'_{\{1\}}, V'_{\{2\}}, V'_{\{1,2\}})$ be a $\gamma_{oir2}(G \odot H)$ -function and let $v_i \in V(G)$. We consider the following cases.

Case 1. $g(v_i) = \emptyset$. Since $V'_\emptyset$ is independent, we have $|g(w)| \geq 1$ for all $w \in V(H_i)$. Therefore, $|g(V(H_i) \cup \{v_i\})| \geq |V(H)|$.

Case 2. $g(v_i) = \{1\}$ or $\{2\}$. Since $V'_\emptyset$ is independent, at most $\alpha(H)$ vertices of H_i can be assigned $\emptyset$ under g . Moreover, the isolated vertices of $H_i \cong H$ cannot be assigned $\emptyset$ under g . Therefore, $|g(w)| \geq 1$ for all $w \in I \setminus I_H$. This implies that $|g(V(H_i) \cup \{v_i\})| \geq |V(H)| - \alpha(H) + i_H + 1 = \beta(H) + i_H + 1$.

Case 3. $g(v_i) = \{1, 2\}$. Note that at most $\alpha(H) = |V(H)| - \beta(H)$ vertices of H_i can be assigned $\emptyset$ under g by a similar fashion. Therefore, $|g(V(H_i) \cup \{v_i\})| \geq \beta(H) + 2$.

On the other hand, since g is an OI2RD function of $G \odot H$, it follows that the function $f''_G = (V''_\emptyset, V''_{\{1\}}, V''_{\{2\}}, V''_{\{1,2\}}) = (V'_\emptyset \cap V(G), V'_{\{1\}} \cap V(G), V'_{\{2\}} \cap V(G), V'_{\{1,2\}} \cap V(G))$ fulfills the independence of $V''_\emptyset = V'_\emptyset \cap V(G)$. As a consequence of all the cases above, we deduce that

$$\begin{aligned} \gamma_{oir2}(G \odot H) = & \sum_{i=1}^n |g(V(H_i) \cup \{v_i\})| \geq |V''_\emptyset||V(H)| + |V''_{\{1\}}|(\beta(H) + i_H + 1) + |V''_{\{2\}}|(\beta(H) + i_H + 1) \\ & + |V''_{\{1,2\}}|(\beta(H) + 2) \\ \geq & \min\{|V_\emptyset||V(H)| + |V_{\{1\}}|(\beta(H) + i_H + 1) + |V_{\{2\}}|(\beta(H) + i_H + 1) + |V_{\{1,2\}}|(\beta(H) + 2)\}, \end{aligned}$$

taken over all possible functions $f_G = (V_\emptyset, V_{\{1\}}, V_{\{2\}}, V_{\{1,2\}})$ of G for which $V_\emptyset$ is independent in G . Therefore,

$$\gamma_{oir2}(G \odot H) = \min\{|V_\emptyset||V(H)| + |V_{\{1\}}|(\beta(H) + i_H + 1) + |V_{\{2\}}|(\beta(H) + i_H + 1) + |V_{\{1,2\}}|(\beta(H) + 2)\},$$

taken over all possible above-mentioned functions f_G .

Clearly, $\beta(H) + 1 \leq \min\{\beta(H) + 2, |V(H)|\}$ when H has no isolated vertices, and $\beta(H) + 2 \leq \min\{\beta(H) + i_H + 1, |V(H)|\}$ otherwise. Taking these facts into consideration, the function f_G for which we get the minimum in the right-hand side of the last equality assigns $\{1\}$ or $\{2\}$ to all vertices of G when H has no isolated vertices, and assigns $\{1, 2\}$ to all vertices of G otherwise. Consequently, $\gamma_{oir2}(G \odot H) = n(\beta(H) + 1)$ when $i_H = 0$, and $\gamma_{oir2}(G \odot H) = n(\beta(H) + 2)$ when $i_H \neq 0$. This completes the proof. $\square$

Acknowledgements. This research has been supported by the Discrete Mathematics Laboratory of the Faculty of Mathematical Sciences at Alzahra University.

REFERENCES

- [1] B. Brešar, M. Henning and D. Rall, Rainbow domination in graphs. *Taiwanese J. Math.* **12** (2008) 213–225.
- [2] A. Cabrera Martínez, A note on outer-independent 2-rainbow domination in graphs. *Mathematics* **10** (2022) 2287.
- [3] T. Gallai, Über extreme Punkt-und Kantenmengen. *Ann. Univ. Sci. Budapest Eötvös Sect. Math.* **2** (1959) 133–138.
- [4] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-completeness. W.H. Freeman & Co., New York, USA (1979).
- [5] C.D. Godsil and B.D. McKay, A new graph product and its spectrum. *Bull. Aus. Math. Soc.* **18** (1978), 21–28.
- [6] R. Hammack, W. Imrich and S. Klavžar, Handbook of Product Graphs, 2nd edition. CRC Press, Boca Raton, FL (2011).
- [7] T.W. Haynes, S.T. Hedetniemi and M.A. Henning, editors. Topics in Domination in Graphs. Springer International Publishing, Switzerland (2020).
- [8] Q. Kang, V. Samodivkin, Z. Shao, S.M. Sheikholeslami and M. Soroudi, Outer-independent k -rainbow domination. *J. Taibah. Univ. Sci.* **13** (2019) 883–891.
- [9] T. Kraner Šumenjak, D.F. Rall and A. Tepeh, Rainbow domination in the lexicographic product of graphs. *Discrete Appl. Math.* **161** (2013) 2133–2141.
- [10] Z. Mansouri and D.A. Mojdeh, Outer independent rainbow dominating functions in graphs. *Opusc. Math.* **40** (2020) 599–615.
- [11] M. Pilipczuk, M. Pilipczuk and R. Škrekovski, Some results on Vizing’s conjecture and related problems. *Discrete Appl. Math.* **160** (2012) 2484–2490.
- [12] A.J. Schwenk, Computing the characteristic polynomial of a graph, in Graphs Combin, edited by R. Bari and F. Harary. Springer, Berlin (1974) 153–172.
- [13] V.G. Vizing, Some unsolved problems in graph theory. *Uspehi Mat. Nauk.* **23** (1968) 117–134.
- [14] D.B. West, Introduction to Graph Theory, 2nd edition. Prentice Hall, USA (2001).

**Please help to maintain this journal in open access!**

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting subscribers@edpsciences.org.

More information, including a list of supporters and financial transparency reports, is available at <https://edpsciences.org/en/subscribe-to-open-s2o>.