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TRANSIENT ANALYSIS OF A BULK STREAM QUEUE WITH ARBITRARILY DISTRIBUTED ARRIVAL INTERVALS

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Abstract. We study the classical infinite buffer single server queueing model with renewal input of customers in batches of random size, having arbitrarily distributed arrival intervals and exponentially distributed service times. Using the technique of supplementary variable and shift operator we derive closed form expression of the time dependent system content distribution in terms of its Laplace transform. The analysis is mainly based on the root-finding technique of the non-linear characteristic equation in terms of the Laplace transform variable. Additionally, using asymptotic properties of Laplace transform, we deduce the corresponding steady-state distribution. We discuss some special cases of the model, thus providing an alternative approach in deriving the transient distribution. We further evaluate certain performance measures and present extensive numerical examples in tabular and graphical form to illustrate the applicability of our theoretical work. The effect of system parameters, interarrival time distribution and traffic intensity on the system behavior is also demonstrated.

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1. Introduction

In the past many researchers have investigated the classical infinite capacity $GI^X/M/1$ queue for its steady-state solution (see Brière and Chaudhry [10], Easton *et al.* [16,17]). In this queueing model the arrivals occur according to the batch renewal arrival process with random batch size and a single server serves the arrivals individually following exponential distribution. While Easton *et al.* [16] determined the analytical results, Brière and Chaudhry [10] and Easton *et al.* [17] dealt with the numerical computation of the queueing system. In this paper, we attempt the analytical as well as numerical solution of the aforementioned queueing model in transient framework.

Most of the active queueing systems used in different areas are often studied under the steady-state condition. But if we are interested to understand the system behavior before it reaches the equilibrium state, or if we do not have the prior knowledge of the time duration that the system will take to reach the equilibrium state, in such cases the study of transient (time-dependent) behavior becomes utmost important. Another situation when the time-dependent analysis is relevant is when the change in the parameter value occurs due to the change in the environment, for example, server breakdown or any sort of service interruption, or after applying a new control mechanism, as because the steady-state situation for the system is no longer maintained. Nevertheless,

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the analysis of the transient behavior of the system is important, but it is more complicated mathematically as compared to the steady-state counterpart, as the former one requires the initial conditions of the system to be considered, which is not relevant for the latter case. Hence the amount of results available for the transient state is significantly less as compared to the steady-state results. Below we give a brief account of the transient work on queueing models.

Initially, the simplest queueing model, *i.e.*, the $M/M/1$ queue with Poisson input and exponential service time distribution was studied by many researchers. Grassmann [19] used three different methods namely, Runge-Kutta, Liou's method and randomization and compared them for the transient solution of the aforementioned queue. Ledermann and Reuter [27] and Champernowne [12] addressed the same problem using spectral method and combinatorial method respectively, whereas Abate and Whitt [1] studied it using the Laplace transform technique. Moreover, Leguesdron *et al.* [28] used the inversion of generating function technique to obtain the transient solution. Along the same direction Parthasarathy [29] provided a simple and direct approach based on modified Bessel functions technique to obtain the system-content probability distribution for the $M/M/1$ queue, whereas Sharma and Bunday [34] developed a series formula for the time-dependent state probabilities of the same model. Parallel to the infinite buffer models, the simple finite buffer $M/M/1/N$ queue was studied by Sharma and Gupta [33] to obtain the closed form expression of the state probabilities. As a further generalization, the corresponding multiserver, $(M/M/c)$ queue and the $M/M/1$ queue with several constraints were considered by many authors, see [3, 15, 21, 23, 30, 35, 36], and the references therein. One may refer to the recent survey paper by Rubino [31] which considers the transient results of several Markovian queueing models with a focus on closed form solutions and numerical techniques.

Over the years the study on the Markovian queues was extended to the queues with generally distributed service time and/or renewal input queues. Benes [7] and Takács [37] studied the $M/G/1$ queue to investigate the customer's waiting time and the server's busy time, whereas Runnenburg [32] used the probabilistic interpretation of generating and moment generating function to derive some known results for the same queue. One may also see the references in the above articles for some related work on $M/G/1$ queue. For the general input and exponential service time distribution, *i.e.*, $GI/M/1$ queue, Conolly [14] used the difference equation technique to obtain the Laplace transform of the time-dependent system-content distribution, Takács [38] dealt with the same while obtaining the distribution of number of services in a busy period and the distribution of number of transitions occurring in the interval $(0, t)$. Chan *et al.* [13] used the boundary value method to determine transient solutions for Markovian queueing networks. Ayyappan and Shyamala [4] explored the time-dependent solution of the $M^X/G/1$ queueing model, which incorporates Bernoulli vacation and balking. Additionally, they utilized the Tauberian property to derive steady-state results from transient solutions. Kempa [24] employed the concept of embedded Markov chain technique to discuss transient analysis of the finite buffer $GI/M/1/N$ queue, while Kempa and Kobielnik [25] expanded this investigation to include single working vacations, obtaining the transform of the time-dependent queue-size distributions. The exploration of renewal input queues was further extended to batch renewal input queues. Bratiychuk and Kempa [9] analyzed the first busy period and first idle time, along with the number of customers served during the first busy period, for the $GI^\theta/G/1$ queue, using a generalization of Korolyuk's method for semi-Markov random walks. Additionally, Bratiychuk and Kempa [8] derived an explicit formula for the distribution of both time dependent and steady-state queue length, for the $GI^\theta/G/1$ queue.

It may be mentioned that the existing work on the time-dependent analysis of queueing systems were mainly focussed on Markovian models, probably because of the ease in dealing with the exponential inter-arrival time distribution of the Poisson arrival process as it possesses memoryless property. However, we can think of situations when arrival occurs into a system after fixed interval of time, like in deterministic distribution, which finds huge application in telecommunication systems, but it does not exhibit memoryless property. Hence it is always relevant to consider a more general (renewal) arrival process, which considers exponential, Erlang, deterministic, hyper-exponential as a special inter-arrival time distribution. Moreover, arrival process seems to be more realistic when it is assumed to occur in batches of random size, thus making batch renewal arrival process more significant.

While we study the $GI^X/M/1$ queue for its transient solution, our main focus is to derive the time-dependent system-content distribution. We initially use the supplementary variable technique to frame the set of equations governing the system by considering the remaining interarrival time of the next batch as the supplementary variable. We further use the shift operator technique to solve the infinite set of partial difference-differential equations in order to obtain the closed-form system-content distribution in terms of the Laplace transform. The analysis is mainly based on the roots of the characteristic equation inside the unit circle which are obtained in terms of the Laplace transform variable. Finally using the theory of difference equations (see Gupta *et al.* [20], Barbhuiya and Gupta [5,6]) we obtain the transformed probability distribution. It may be mentioned that the methodology used in this paper requires the assumption that the maximum permissible size of the arriving batch should be finite, which is quite reasonable in real world scenarios. Further, to make the theoretical results applicable to the practitioners, we demonstrate the inversion of the highly non-linear Laplace transform expressions using Gaver- Stehfest algorithm (see Kuznetsov [26]) and present the results in tabular and graphical forms. A few major contribution of the work done in this paper is mentioned below.

(1) It provides an elegant methodology for the transient analysis of the infinite buffer $GI^X/M/1$ queue, which can also be extended to study a large class of infinite capacity queueing models.
(2) It generalizes some of the already existing work in the literature and hence yields an alternative solution procedure for them, as discussed in one of the following sections.
(3) It provides the steady-state result of the aforementioned queue, which are derived directly from the transient result, using the asymptotic properties of the Laplace transform.
(4) Through numerical examples we study the impact of interarrival time distributions and the input parameters on the system performance.
(5) We address the case when the interarrival time distribution have a non-rational or a transcendental form of the Laplace-Stieltjes transform. Also, we consider the fact that the arriving batch size may have an infinite support, and hence discuss the way to handle such cases.

The queueing system explored in this paper has potential applications in modeling traffic on high-speed networks, particularly those handling real-time communication such as the transmission of Motion Picture Experts Group (MPEG) data. MPEG utilizes a coding scheme that involves compressing video traffic of a sequence of frames which are categorized as I-frames, P-frames, and B-frames. These frames, forming variable-sized batches known as a Group of Pictures (GOP), are efficiently compressed for transmission and decompressed accurately and quickly. In this system, the sequence of frames awaiting compression may be treated as batch arrivals with varying sizes where the arrival time between any two successive batches may have a general distribution. On the other hand, the compression of these frames by one or more compressor can be thought of as service provided before re-transmission through a proper channel or a transmitter. The analysis of such a queueing model thus play a critical role in optimizing network resources and ensuring quality of service, especially in streaming video applications. It assists network engineers in making informed decisions to maintain efficient data flow and improve network performance (see Takagi and Wu [39]).

The remaining portion of the paper is organized as follows. Section 2 describes the model, Section 3 gives the analytical procedure for the transient solution, Section 4 provides the steady-state distribution from the corresponding transient distribution, Section 5 discusses some special cases of the model, Section 6 evaluates some performance measures, whereas Section 7 demonstrates the numerical validity of the obtained results. Finally it concludes with Section 8.

2. Model description

The model has been described in the literature (see Brière and Chaudhry [10], Easton *et al.* [16,17]), but we briefly present it here just for the sake of completeness.

– The independent and identically distributed (iid) continuous random variable, say T, represents the inter-arrival time of successive arrivals which follow a general independent distribution with cumulative distribu-

tion function (cdf) $A(t)$, probability density function (pdf) $a(t)$, Laplace-Stieltjes transform (LST) $A^*(\theta)$, and the mean interarrival time $a_1 = \frac{1}{\lambda} = -A^{*(1)}(0)$, where $A^{*(1)}(0)$ is the derivative of LST $A^*(\theta)$ at $\theta = 0$.
- Arrivals occur in batches of random size X with probability mass function (pmf) $P(X = j) = g_j$, $j = 1, 2, 3..., l$, where $l \in \mathbb{N}$ is the maximum limit on the arriving batch size.
- There is a single serving point where one customer is served at a time, according to exponential service time distribution, based on a first-come-first-served discipline.
- There is no limit on the queueing capacity of the system; moreover, the inter-arrival time and service time are independent of each other.

3. The analysis

In this section, we first derive the governing equations of the model by means of supplementary variable technique. The infinite set of equations is then solved for the time-dependent probability distribution using Laplace transform and difference equation technique (see Elaidy [18]), for which the following notations are introduced at time t.

- $N(t)$: number of customers in the system
- $R(t)$: remaining inter-arrival time of the next arriving batch
- $p_k(t) = P(N(t) = k), \quad k \geq 0$
- $p_k(r, t)dr = P(N(t) = k, r < R(t) \leq r + dr), \ r \geq 0, \ k \geq 0$

We describe the state of the system at time t as $(N(t), R(t))$ and consider $R(t)$ as the supplementary variable, which restores the Markovian nature of the system. Thus relating the system state at two consecutive time instants t and $t + \Delta t$, we formulate the following governing equations.

$$p_0(r - \Delta r, t + \Delta t) = p_0(r, t) + \mu p_1(r, t)\Delta t + o(\Delta t), \tag{1}$$

$$p_k(r - \Delta r, t + \Delta t) = (1 - \mu\Delta t)p_k(r, t) + \mu p_{k+1}(r, t)\Delta t + a(r)\sum_{i=1}^{k} g_i p_{k-i}(0, t)\Delta t + o(\Delta t), \quad 1 \leq k \leq l - 1 \tag{2}$$

$$p_k(r - \Delta r, t + \Delta t) = (1 - \mu\Delta t)p_k(r, t) + \mu p_{k+1}(r, t)\Delta t + a(r)\sum_{i=1}^{l} g_i p_{k-i}(0, t)\Delta t + o(\Delta t), \quad k \geq l. \tag{3}$$

Utilizing Taylor series expansion on the left-hand side of equations (1)–(3), and then considering $\lim \Delta t \to 0$, yields the following set of partial difference-differential equations.

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\Big)p_0(r, t) = \mu p_1(r, t), \tag{4}$$

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\Big)p_k(r, t) = -\mu p_k(r, t) + \mu p_{k+1}(r, t) + a(r)\sum_{i=1}^{k} g_i p_{k-i}(0, t), 1 \leq k \leq l - 1 \tag{5}$$

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial r}\Big)p_k(r, t) = -\mu p_k(r, t) + \mu p_{k+1}(r, t) + a(r)\sum_{i=1}^{l} g_i p_{k-i}(0, t), \ k \geq l. \tag{6}$$

We use the Laplace transforms as defined below.

$$P_k^*(\theta, \phi) = \int_0^\infty \int_0^\infty p_k(r, t)e^{-\theta r}e^{-\phi t}drdt,$$

$$Q_k^*(\phi) = \int_0^\infty p_k(0,t)e^{-\phi t}dt \quad \text{and}$$

$$A^*(\theta) = \int_0^\infty a(r)e^{-\theta r}dr, \quad \text{where} \quad \Re(\theta) \geq 0, \Re(\phi) \geq 0$$

The symbols $\Re(\theta)$ and $\Re(\phi)$ represents the real part of the respective transform variable. Using the above transforms and the initial conditions $p_0(r,0) = a(r)$ and $p_k(r,0) = 0$ for $k \geq 1$, equations (4)–(6) reduces to

$$(\phi - \theta)P_0^*(\theta,\phi) + Q_0^*(\phi) = A^*(\theta) + \mu P_1^*(\theta,\phi) \tag{7}$$

$$(\phi + \mu - \theta)P_k^*(\theta,\phi) + Q_k^*(\phi) = \mu P_{k+1}^*(\theta,\phi) + A^*(\theta)\sum_{i=1}^{k} g_i Q_{k-i}^*(\phi), \quad 1 \leq k \leq l-1 \tag{8}$$

$$(\phi + \mu - \theta)P_k^*(\theta,\phi) + Q_k^*(\phi) = \mu P_{k+1}^*(\theta,\phi) + A^*(\theta)\sum_{i=1}^{l} g_i Q_{k-i}^*(\phi), \quad k \geq l. \tag{9}$$

Adding equations (7)–(9) for all values of k, we have

$$(\phi - \theta)\sum_{k=0}^{\infty} P_k^*(\theta,\phi) = (A^*(\theta) - 1)\sum_{k=0}^{\infty} Q_k^*(\phi) + A^*(\theta). \tag{10}$$

Putting $\theta = \phi$ in the above equation gives

$$\sum_{k=0}^{\infty} Q_k^*(\phi) = \frac{A^*(\phi)}{1 - A^*(\phi)}\cdot \tag{11}$$

In the forthcoming section we employ the difference equation technique to obtain the explicit solution for the transient probability distribution.

3.1. The transient distribution

In order to get the transient distribution we initially apply the right shift operator D in equation (9), which is defined as $D^n P_k^*(\theta,\phi) = P_{n+k}^*(\theta,\phi)$ and $D^n Q_k^*(\phi) = Q_{n+k}^*(\phi)$, where $n \geq 1, k \geq 0$ and it reduces to

$$(\phi - \theta + \mu - \mu D)P_k^*(\theta,\phi) = \left[A^*(\theta)\sum_{i=1}^{l} g_i D^{l-i} - D^l\right] Q_{k-l}^*(\phi), \quad k \geq l. \tag{12}$$

Replacing θ by $(\phi + \mu - \mu D)$ in (12), we obtain the homogeneous linear difference equation with constant coefficient as

$$\left[A^*(\phi + \mu - \mu D)\sum_{i=1}^{l} g_i D^{l-i} - D^l\right] Q_k^*(\phi) = 0, \quad k \geq 0. \tag{13}$$

The corresponding auxiliary equation is given by

$$A^*(\phi + \mu - \mu z)\sum_{i=1}^{l} g_i z^{l-i} - z^l = 0. \tag{14}$$

Taking note of the convergence of $Q_k^*(\phi)$, we consider only those roots of equation (14) which lie inside the unit circle. Using Rouché's theorem (see Brown and Churchill [11]) the number of inside roots of equation (14) can be easily determined. For $f(z) = z^l$, $g(z) = A^*(\phi + \mu - \mu z)\sum_{i=1}^{l} g_i z^{l-i} = K(z)\sum_{i=1}^{l} g_i z^{l-i}$, $|z| = 1$ and $\Re(\phi) > 0$ we have

$$|f(z)| = 1 \text{ and } |g(z)| = |K(z)||\sum_{i=1}^{l} g_i z^{l-i}| \le K(|z|)\sum_{i=1}^{l} g_i |z|^{l-i} = A^*(\phi) < 1 = |f(z)|.$$

Hence equation (14) will have exactly l roots, denoted by γ_i, $i = 1, 2, \ldots, l$ inside the unit circle which are in terms of ϕ. Consequently, the general solution of equation (13) is of the form

$$Q_k^*(\phi) = \sum_{j=1}^{l} \kappa_j \gamma_j^k, \quad k \ge 0, \tag{15}$$

where κ_i's are the corresponding unknown coefficients of γ_i's, which are also in terms of ϕ but independent of k. Now substituting (15) in (12), we obtain

$$(\phi + \mu - \mu D - \theta)P_k^*(\theta, \phi) = A^*(\theta)\sum_{i=1}^{l} g_i \sum_{j=1}^{l} \kappa_j \gamma_j^{k-i} - \sum_{j=1}^{l} \kappa_j \gamma_j^k, \quad k \ge l, \tag{16}$$

which is a non-homogeneous linear difference equation with constant coefficient. Its general solution is given by the sum of the solution to its homogeneous part $(P_{kh}^*(\theta, \phi))$ and a particular solution $(P_{kp}^*(\theta, \phi))$. The homogeneous solution is of the form $P_{kh}^*(\theta, \phi) = C(z(\theta, \phi))^k$ where C is a constant independent of k and $z(\theta, \phi) = \left\{\frac{(\phi - \theta)}{\mu} + 1\right\}$ is a root of the auxiliary equation corresponding to the homogeneous part. Correspondingly, the particular solution is given by

$$P_{kp}^*(\theta, \phi) = \sum_{j=1}^{l} \kappa_j \left\{\frac{1 - A^*(\theta)\sum_{i=1}^{l} \gamma_j^{-i} g_i}{\theta - \phi - \mu + \mu\gamma_j}\right\}\gamma_j^k, \quad k \ge l,$$

and hence the general solution of (16) is

$$P_k^*(\theta, \phi) = C(z(\theta, \phi))^k + \sum_{j=1}^{l} \kappa_j \left\{\frac{1 - A^*(\theta)\sum_{i=1}^{l} \gamma_j^{-i} g_i}{\theta - \phi - \mu + \mu\gamma_j}\right\}\gamma_j^k, \quad k \ge l. \tag{17}$$

Upon applying the limit as $\theta \to 0$ in equation (10), we obtain $\sum_{k=0}^{\infty} P_k^*(0, \phi) = \frac{1}{\phi}$, which is the Laplace transform of the normalizing condition $\sum_{k=0}^{\infty} p_k(t) = 1$. Hence, we must have

$$\sum_{k=l}^{\infty} P_k^*(0, \phi) \le \frac{1}{\phi}\cdot \tag{18}$$

Applying the same limit in equation (17) and summing over k from l to ∞ we get the first term in the right-hand side of (17) as $C\sum_{k=l}^{\infty}\left\{\left(\frac{\phi}{\mu} + 1\right)\right\}^k \longrightarrow \infty$. Thus, to maintain the convergence we must have the constant $C = 0$, which reduces equation (17) to

$$P_k^*(\theta, \phi) = \sum_{j=1}^{l} \kappa_j \left\{\frac{1 - A^*(\theta)\sum_{i=1}^{l} \gamma_j^{-i} g_i}{\theta - \phi - \mu + \mu\gamma_j}\right\}\gamma_j^k, \quad k \ge l. \tag{19}$$

In the above expression, the only unknowns are the κ_j's, $j = 1, 2, \ldots, l$ which are functions of ϕ and are to be determined. Now considering that (19) is true for $1 \le k \le l - 1$ as well, we substitute it in equation (8) to get

$$\sum_{j=1}^{l} \kappa_j \sum_{i=k+1}^{l} \gamma_j^{k-i} g_i = 0, \quad 1 \le k \le l - 1. \tag{20}$$

Taking values for $k = l-1, l-2, \ldots, 1$, in (20) we will have $l-1$ homogeneous equations as

$$\sum_{j=1}^{l} \frac{\kappa_j}{\gamma_j} = 0, \sum_{j=1}^{l} \frac{\kappa_j}{\gamma_j^2} = 0, \ldots \sum_{j=1}^{l} \frac{\kappa_j}{\gamma_j^{l-1}} = 0. \tag{21}$$

Also, using (15) in (11), we get

$$\sum_{j=1}^{l} \frac{\kappa_j}{1-\gamma_j} = \frac{A^*(\phi)}{1-A^*(\phi)}. \tag{22}$$

Equations (21) and (22) together can be used to find the unknown κ_j's, $j = 1, 2, \ldots, l$ in terms of ϕ and hence we obtain $P_k^*(\theta, \phi)$ in terms of known quantities as

$$P_k^*(\theta, \phi) = \sum_{j=1}^{l} \kappa_j \left\{ \frac{1 - A^*(\theta) \sum_{i=1}^{l} \gamma_j^{-i} g_i}{\theta - \phi - \mu + \mu\gamma_j} \right\} \gamma_j^k, \quad k \geq 1. \tag{23}$$

Further, using equation (11) in (10), we get

$$\sum_{k=0}^{\infty} P_k^*(\theta, \phi) = \left\{ \frac{A^*(\theta) - A^*(\phi)}{(\phi - \theta)(1 - A^*(\phi))} \right\} \tag{24}$$

and hence using (23) we can deduce $P_0^*(\theta, \phi)$ as

$$P_0^*(\theta, \phi) = \left\{ \frac{A^*(\theta) - A^*(\phi)}{(\phi - \theta)(1 - A^*(\phi))} \right\} - \sum_{j=1}^{l} \kappa_j \left\{ \frac{1 - A^*(\theta) \sum_{i=1}^{l} \gamma_j^{-i} g_i}{\theta - \phi - \mu + \mu\gamma_j} \right\} \frac{\gamma_j}{1-\gamma_j}. \tag{25}$$

Now letting $\theta = 0$ in (23) and (25) we have

$$P_k^*(0, \phi) = \sum_{j=1}^{l} \kappa_j \left\{ \frac{1 - \sum_{i=1}^{l} \gamma_j^{-i} g_i}{\mu\gamma_j - \phi - \mu} \right\} \gamma_j^k, \quad k \geq 1, \tag{26}$$

$$P_0^*(0, \phi) = \frac{1}{\phi} - \sum_{j=1}^{l} \kappa_j \left\{ \frac{1 - \sum_{i=1}^{l} \gamma_j^{-i} g_i}{\mu\gamma_j - \phi - \mu} \right\} \frac{\gamma_j}{1-\gamma_j}. \tag{27}$$

Equations (26) and (27) gives the closed form expression of the Laplace transform of the transient distribution $p_k(t), k \geq 0$. As a matter of fact, the Laplace transform of $p_0(t)$ can also be obtained by using equation (23) for $k = 1$ in equation (7) and then considering $\theta = 0$. It may be mentioned that as the transforms are obtained in a completely known form, they can be inverted numerically to obtain the time-dependent probabilities, as illustrated in the following section and also in Section 7.

3.2. The computation steps

In this section, we outline a step-by-step procedure for numerically obtaining the transient probabilities. It also describes the Gaver-Stehfest algorithm (see Kuznetsov [26]) which is essential to numerically invert the Laplace transform. The steps are as follows.

(1) Provide fixed value for the input parameters λ, μ, l, $A^*(s)$ and g_i for all $i = 1, 2, \ldots, l$.
(2) Solve equation (14) to find the l roots, γ_j for all $j = 1, 2, \ldots, l$, in terms of ϕ, within the unit circle.
(3) Solve the system of equations given in (21) and (22) to determine the unknown coefficients κ_j for all $j = 1, 2, \ldots, l$.

(4) Obtain the Laplace transform of the transient distribution given in (26) and (27) for all k.
(5) Utilize the Garver-Stehfest algorithm (described below) to invert the Laplace transform.

Consider $f : (0, \infty) \longrightarrow \mathbb{R}$ as a locally integrable function with its Laplace transform denoted by $F(s)$. We numerically invert $F(s)$ to obtain $f(x)$ using the Gaver-Stehfest algorithm. It is an iterative algorithm which converges towards $f(x)$ through a sequence of functions,

$$f_n(x) = \frac{\ln 2}{x} \sum_{k=1}^{2n} \left(a_k(n) F\left(\frac{k \ln 2}{x}\right)\right)$$

where the coefficients $a_k(n)$ are determined by,

$$a_k(n) = (-1)^{n+k} \sum_{j=\lfloor \frac{k+1}{2} \rfloor}^{min(n,k)} j^{n+1} \binom{n}{j} \binom{2j}{j} \binom{j}{k-j}, \quad n \geq 1, 1 \leq k \leq 2n,$$

and $\lfloor . \rfloor$ is the greatest integer function. It is to be noted that $F(s)$ must be replaced by $P_k^*(0, \phi)$in order to get $p_k(t)$.

4. The steady-state distribution

In Section 3, we have obtained $p_k(t)$ in terms of its Laplace transform. In this section, we derive the steady-state distribution directly from these results, as illustrated below. By solving equations (21) and (22) we get κ_j in terms of $\gamma_j' s$ as

$$\kappa_j = \frac{(-1)^l A^*(\phi) \gamma_j^{l-1} \prod_{m=1}^{l} (\gamma_m - 1)}{(1 - A^*(\phi)) \prod_{m=1, m\neq j}^{l} (\gamma_j - \gamma_m)}, \quad j = 1, 2, \ldots, l. \tag{28}$$

We can deduce the steady-state probabilities using the asymptotic property of Laplace transform (see Conolly [14], Parthasarathy and Sharafali [30]) given by

$$p_k = \lim_{t\to\infty} p_k(t) = \lim_{\phi\to 0} \phi P_k^*(0, \phi), \quad k \geq 0. \tag{29}$$

Substituting equations (28) in (26) and (27) and further using (29) we obtain,

$$p_k = \lim_{\phi\to 0} \phi \sum_{j=1}^{l} \left\{ \frac{\left((-1)^l A^*(\phi) \gamma_j^{l-1} \prod_{m=1}^{l} (\gamma_m - 1)\right) \left(1 - \sum_{i=1}^{l} \gamma_j^{-i} g_i\right) \gamma_j^k}{(1 - A^*(\phi)) \prod_{m=1, m\neq j}^{l} (\gamma_j - \gamma_m) (\mu\gamma_j - \phi - \mu)} \right\}, \quad k \geq 1$$
$$p_0 = \lim_{\phi\to 0} \phi \left\{ \frac{1}{\phi} - \sum_{j=1}^{l} \left\{ \frac{\left((-1)^l A^*(\phi) \gamma_j^{l-1} \prod_{m=1}^{l} (\gamma_m - 1)\right) \left(1 - \sum_{i=1}^{l} \gamma_j^{-i} g_i\right) \gamma_j}{(1 - A^*(\phi)) \prod_{m=1, m\neq j}^{l} (\gamma_j - \gamma_m) (\mu\gamma_j - \phi - \mu) (1 - \gamma_j)} \right\} \right\},$$

which after simplification yields

$$p_k = \sum_{j=1}^{l} \frac{G_j (1 - \sum_{i=1}^{l} \gamma_j(0)^{-i} g_i) \gamma_j(0)^k}{\mu(\gamma_j(0) - 1)}, \quad k \geq 1$$
$$p_0 = 1 - \sum_{j=1}^{l} \frac{G_j (\sum_{i=1}^{l} \gamma_j(0)^{-i} g_i) \gamma_j(0)}{(\mu(\gamma_j(0) - 1))(1 - \gamma_j(0))}.$$

Here $G_j = \dfrac{(-1)^l \lambda \gamma_j(0)^{l-1} \prod_{m=1}^{l} (\gamma_m(0) - 1)}{\prod_{m=1, m\neq j}^{l} (\gamma_j(0) - \gamma_m(0))}$ and $\gamma_j(0)$ are the roots at $\phi = 0$, for all $j = 1, 2 \ldots, l$. These results align with the existing results (see Barbhuiya and Gupta [5]) for the steady-state distribution of $GI^X/M/1$ queue.

5. Special cases

In this section, we deduce the transient result for some special cases of the model by taking fixed values of certain parameters. Since these models are already studied in the literature, our analysis provides an alternative procedure to obtain the solution.

– The results of the $GI/M/1$ queue (see Conolly [14]) can be derived by considering $g_1 = 1$ and $g_i = 0$, $\forall$ $i \neq 1$, *i.e.*, $l = 1$. Therefore the Laplace transform of the distributions will reduce to

$$P_k^*(0, \phi) = \frac{(1-\gamma_1)^2 \gamma_1^{k-1} A^*(\phi)}{(1 - A^*(\phi))(\phi + \mu - \mu\gamma_1)} \quad k \geq 1, \tag{30}$$

$$P_0^*(0, \phi) = \frac{1}{\phi} - \frac{(1-\gamma_1) A^*(\phi)}{(1 - A^*(\phi))(\phi + \mu - \mu\gamma_1)}, \tag{31}$$

where γ_1 is the only root of $z = A^*(\phi + \mu - \mu z)$ inside the unit circle and is a function of ϕ.

– Similarly, for the simplest Markovian queueing model, *i.e.*, $M/M/1$ queue $g_1 = 1$ and $g_i = 0$, $\forall$ $i \neq 1$ and $A^*(\theta) = \dfrac{\lambda}{\lambda + \theta}$, where λ is the rate at which customers arrive. Thus the probability distributions in terms of its Laplace transform are given by

$$P_k^*(0, \phi) = \frac{\lambda(1-\gamma_1)^2 \gamma_1^{k-1}}{\phi(\phi + \mu - \mu\gamma_1)} \quad k \geq 1, \tag{32}$$

$$P_0^*(0, \phi) = \frac{1}{\phi} - \frac{\lambda(1-\gamma_1)}{\phi(\phi + \mu - \mu\gamma_1)}, \tag{33}$$

where $\gamma_1 = \dfrac{\phi + \lambda + \mu - \sqrt{(\phi + \lambda + \mu) - 4\lambda\mu}}{2\mu}$ is the only root of $z = A^*(\phi + \mu - \mu z)$ inside the unit circle. It has been verified numerically that the results tally with the existing results in the literature (see Kashyap and Chaudhry [22]) as illustrated in Section 7.

6. Performance measures

In this section, the performance characteristics of the system is evaluated directly from the known distribution. Although the results are obtained in terms of its Laplace transform, they can be inverted to get the exact time-dependent measures. The expected number of customers in the system at any time t is given by $E(N(t)) = \sum_{k=0}^{\infty} k p_k(t)$ and hence the Laplace transform of the expected number of customers in the system will be

$$E^*(\phi) = \sum_{j=1}^{l} \kappa_j \left\{ \frac{1 - \sum_{i=1}^{l} \gamma_j^{-i} g_i}{\mu\gamma_j - \phi - \mu} \right\} \frac{\gamma_j}{(1-\gamma_j)^2}. \tag{34}$$

Similarly, we can compute the variance of the number of customers in the system at time t. The variance, denoted as $V(N(t))$, is determined by subtracting $(E(N(t)))^2$ form $E(N(t)^2)$. To numerically evaluate this, we need to invert the Laplace transform of $E(N(t)^2)$ and $(E(N(t)))^2$. The Laplace transform of the former one is given by

$$E_1^*(\phi) = \sum_{j=1}^{l} \kappa_j \left\{ \frac{1 - \sum_{i=1}^{l} \gamma_j^{-i} g_i}{\mu\gamma_j - \phi - \mu} \right\} \frac{\gamma_j(1+\gamma_j)}{(1-\gamma_j)^3}$$

TABLE 1. Distribution of the number of customers in the system at different time instant t for an $M/M/1$ queue.

k\t	1	10	20	30	40	50	100	150	200	250
$P_0(t)$	0.6338	0.50328	0.50025	0.50002	0.5	0.5	0.5	0.5	0.5	0.5
$P_1(t)$	0.25728	0.25117	0.25009	0.25001	0.25	0.25	0.25	0.25	0.25	0.25
$P_2(t)$	0.0823	0.12495	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125
$P_3(t)$	0.0211	0.06186	0.06246	0.06249	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
$P_4(t)$	0.00445	0.03042	0.03119	0.03124	0.03124	0.03125	0.03125	0.03125	0.03125	0.03125
$P_5(t)$	0.00079	0.01483	0.01556	0.01561	0.01562	0.01562	0.015625	0.015625	0.015625	0.015625
$P_6(t)$	0.00012	0.00715	0.00776	0.007806	0.007811	0.007812	0.007812	0.007813	0.007813	0.007813
$P_7(t)$	0.000016	0.00334	0.00386	0.003901	0.003905	0.003906	0.003906	0.003906	0.003906	0.003906
$P_8(t)$	0.000002	0.00159	0.00192	0.00195	0.001952	0.001953	0.001953	0.001953	0.001953	0.001953
$P_9(t)$	0.000000	0.00073	0.00095	0.00097	0.000976	0.000976	0.000976	0.000977	0.000977	0.000977
$P_{10}(t)$	0.000000	0.00033	0.00047	0.000486	0.000488	0.000488	0.000488	0.000488	0.000488	0.000488
.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.
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Sum	1.000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

whereas the Laplace transform of the latter one is given in equation (34). The n^{th} moment of $N(t)$ is derived as

$$m_n = (-1)^n n! \left[\sum_{j=1}^{l} \kappa_j \left\{ \frac{1 - \sum_{i=1}^{l} \gamma_j^{-i} g_i}{\mu^{n+1}(\gamma_j - 1)^{n+1}} \right\} \gamma_j^k \right]. \tag{35}$$

which can be easily proved using the method of mathematical induction by taking, $m_n = (-1)^n P_k^{*(n)}(0,0)$, where $P_k^{*(n)}(0,0)$ is the n^{th} derivative of $P_k^*(0,\phi)$ calculated at $\phi = 0$.

7. Numerical analysis

Till now we have obtained the transient probability distributions and the corresponding performance measures in terms of its Laplace transform. From a practitioners point of view, the distributions expressed in this form are of very little importance. Motivated by this, in this section we use the Gaver- Stehfest algorithm (see Kuznetsov [26]), to numerically invert the Laplace transform and hence we generate a couple of results which are presented in tabular and graphical forms. The computations are done using Mathematica 13 software in a desktop with 12^{th} Gen Intel Core i5 Processor with 16 GB RAM.

For simplicity and easy comparison purpose, we have initially considered the single arrival $GI/M/1$ queue with $GI = M$ and other parameters are taken as $\lambda = 1$, $\mu = 2$. We have inverted the Laplace transform given in (32) and (33) in order to get the transient probabilities of the system size. The results are presented in Table 1. The first horizontal row represents several time points at which the probabilities $p_k(t)$ are computed. When t is kept fixed, it may be observed that the probability values add up to 1, for any value of t, which should be the

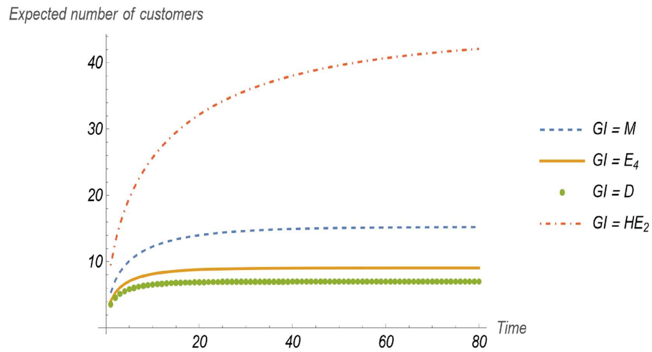

FIGURE 1. Expected system-content ($L(t)$) for different inter-arrival time distributions.

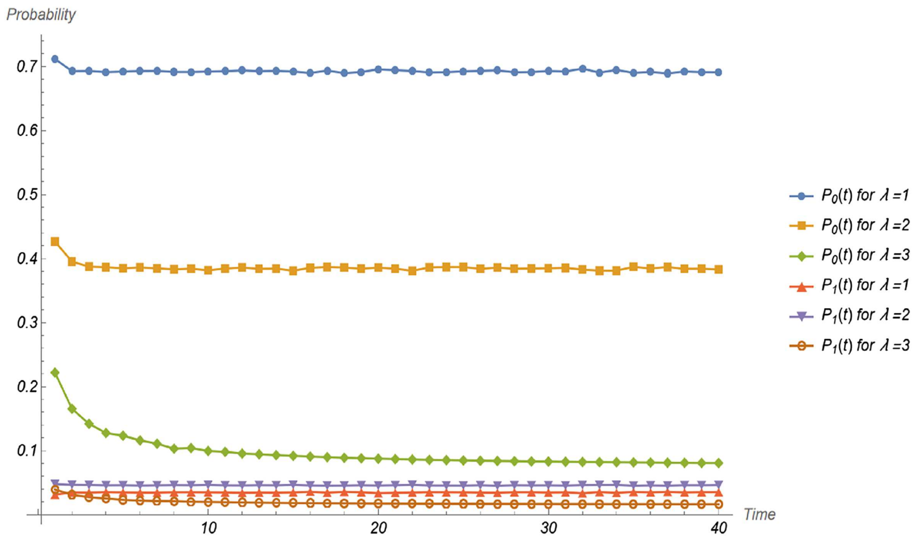

FIGURE 2. Effect of arrival rate (λ) on the transient probability ($p_k(t)$) for $k = 0$ and $k = 1$.

case, since $\sum_{n=0}^{\infty} p_n(t) = 1$ for a fixed t. While, as one moves horizontally for a given k, the probability values initially fluctuates and then simultaneously converges to a constant term, which is the steady-state probability and is equal to $(1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^k$ for an $M/M/1$ queue. This observation is true for all values of k and is in accordance with the known theory that after certain time instant, the system attains a steady-state for $\frac{\lambda}{\mu} < 1$. It may be mentioned that we have checked the numerical values in Table 1 with the existing results in the literature (see Kashyap and Chaudhry [22]) and found to be exactly the same. This table may also be used by researchers in future as a reference for cross-checking their results, if needed. Further, in order to study the effect of interarrival time distribution on the system performance, we have plotted the expected number of customers in the system ($L(t)$) against different time instants (t) for exponential ($GI = M$), Erlang ($GI = E_4$), deterministic ($GI = D$) and hyperexponential ($GI = HE_2$) distributions (see Fig. 1). The parameters are chosen as $\lambda = 9$ ($\lambda_1 = 3$, $\lambda_2 = 63$, $\alpha_1 =$, $\alpha_2 =$ for $GI = HE_2$), $\mu = 23$, $g_1 = 0.37$, $g_2 = 0.26$, $g_3 = 0.19$, $g_4 = 0.12$, $g_5 = 0.04$, $g_6 = 0.02$. It may be observed that $L(t)$ is almost the same for D, E_4 and M but it is different for HE_2 at the very beginning. As time increases, the difference in $L(t)$ for all the four distributions become significant and follows the trend $L_{HE_2}(t) > L_M(t) > L_{E_4}(t) > L_D(t)$ for any fixed time t. Also, the difference between $L(t)$ for D and

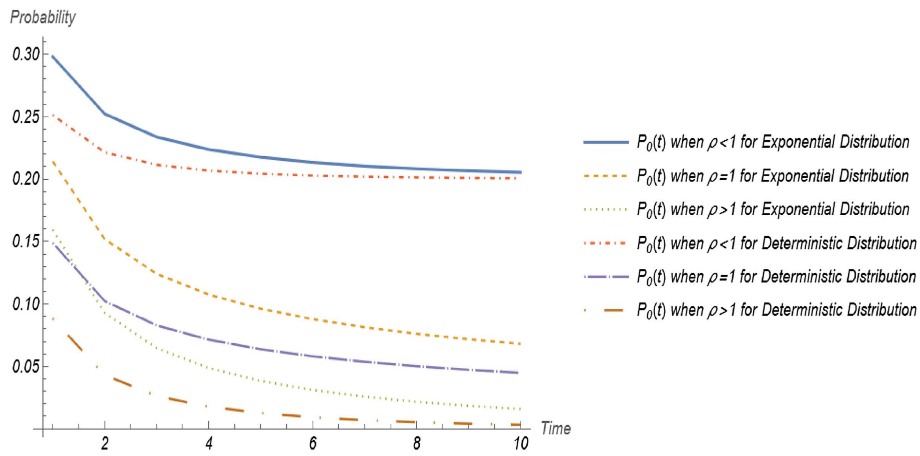

FIGURE 3. Effect of traffic intensity (ρ) on $p_0(t)$ with exponential and deterministic inter-arrival time distribution.

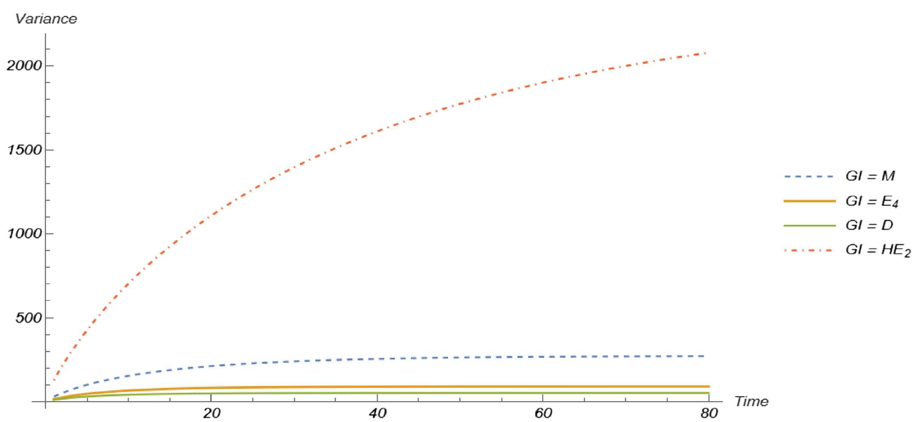

FIGURE 4. Effect of inter-arrival time distribution on variance.

E_4 is comparatively less than the other differences, *i.e.*, $L_{E_4}(t) - L_D(t) < L_M(t) - L_{E_4}(t) < L_{HE_2}(t) - L_M(t)$ for a given t. Moreover, as one moves horizontally along each of the curves, we see that the curves become parallel to the time axis after a certain instant. This signifies that $L(t)(=\sum_{n=0}^{\infty} np_n(t))$ becomes constant and attains a steady-state as time increases, which is because $p_n(t)$ becomes independent of time as t increases, for the traffic intensity $\rho(=\frac{\lambda\bar{g}}{\mu}) < 1$. However, the time taken to reach the steady-state differs significantly among all the four distributions, with deterministic interarrival time distribution taking the least (approximately 10 unit time) whereas the hyperexponential distribution taking the highest time (approximately 80 unit time). Henceforth, with the representation in Figure 1 we may conclude that the interarrival time distribution plays a major role in determining the system performance and should be taken into account. It may be mentioned that we have considered the case when the Laplace transform of the interarrival time distribution is not a rational function and is in a transcendental form (for $GI = D$, $A^*(s) = e^{-\frac{s}{\lambda}}$). To handle such situations we have used Padé(10,10) which gives a rational approximation to the transcendental function (see Akar and Arikan [2]).

In Figure 2 we have shown the effect of arrival rate (λ) on the transient probability distribution. We have assumed $\lambda = 1, 2$ and 3, and $k = 0$ and 1 for $p_k(t)$. The interarrival time distribution is chosen as Erlang (E_3)

and $\mu = 26$. Since our analysis assumes the maximum size of the arriving batch to be finite, in this example we show how to handle the infinite batch size situation. For Figure 2 we have considered batch size (X) to follow Poisson distribution. To handle this, we have taken $g_i = \frac{e^{-p}p^i}{i!\sum_{n=1}^{l}\frac{e^{-p}p^n}{n!}}$, $i = 1, 2, \ldots, l$, where $l = 20$ and $p = 8$ which is truncated Poisson distribution. So whenever we encounter batch size distribution with infinite support we can use the corresponding truncated distribution by truncating it to some large value. Now in Figure 2 it may be observed that after certain time instant t the transient probability attains the steady-state behavior for both $k = 0$ and 1 and for any value of λ. However, for a fixed t, $p_0(t)$ is higher for $\lambda = 1$ as compared to $\lambda = 2$ and 3. It is because with the increase in λ, the traffic intensity increases, hence the probability that the system will remain empty at any time instant will certainly decrease. At the same time, the difference in $p_1(t)$ for different values of λ is very minute, though it is the highest for $\lambda = 2$ as compared to $\lambda = 1$ and 3. Also, one may note that in comparison to $\lambda = 1$ and 2, for $\lambda = 3$ both $p_0(t)$ and $p_1(t)$ shows the transient behavior for a longer period of time before attaining the steady-state.

In Figure 3, we illustrate the influence of traffic intensity, denoted by $\rho(= \frac{\lambda\bar{g}}{\mu})$, on the time-dependent probability $p_0(t)$ for exponential and deterministic interarrival time distribution across different values of ρ $(\leq 1, = 1$ and $\geq 1)$. We set $\lambda = 6$, $g_1 = 0.4$, $g_2 = 0.3$, $g_3 = 0.2$, $g_4 = 0.1$ and $\mu = 15, 12, 10$ corresponding to $\rho < 1$, $\rho = 1$, and $\rho > 1$. For $\rho < 1$, $p_0(t)$ decreases gradually and then stabilizes, becoming time-independent whereas for $\rho = 1$ and $\rho > 1$, it converges to zero. This behavior is observed for both the interarrival time distributions. Notably, for $\rho > 1$, the probability converges to zero more rapidly than for $\rho = 1$. These results hold for any value of k within its range. Additionally, it can be noted that the probabilities are higher for exponential distribution compared to the deterministic ones at any fixed time t.

In Figure 4, we examine the impact of interarrival time distribution distribution on the variance of the system content. We consider exponential (M), Erlang (E_4), deterministic (D) and hyper-exponential (HE_2) as the arrival patterns. The parameters are chosen as $\lambda = 9$, $\mu = 23$, $l = 6$, $g_1 = 0.37$, $g_2 = 0.26$, $g_3 = 0.19$, $g_4 = 0.12$, $g_5 = 0.04$, and $g_6 = 0.02$. Additionally, we set $\alpha_1 = 0.3$, $\alpha_2 = 0.7$, $\lambda_1 = 3$, and $\lambda_2 = 63$ for the hyper-exponential distribution. The horizontal axis spans from 0 to 80, representing time, while the vertical axis, ranging from 0 to 2000, represents variance. The graph indicates that variance stabilizes faster for deterministic, Erlang and exponential interarrival time distributions compared to the hyper-exponential distribution. Furthermore, variance is lowest for deterministic distribution, slightly higher for Erlang, higher for exponential, and highest for hyper-exponential distribution. Notably, the difference in variance between deterministic and Erlang distributions is smaller than that between Erlang and exponential distributions, while the difference in variance between exponential and hyper-exponential distributions is most significant. Variance remains below 200 for exponential, deterministic and Erlang arrival patterns, whereas it surpasses 2000 and stabilizes beyond for hyper-exponential distribution. Similar to Figure 1, from Figure 4 we can infer that the interarrival time distribution plays a crucial role in determining the system performance.

So far we have demonstrated the practical utility of the theoretical results through some numerical examples. These numerical implementations are mainly based on the inversion of the Laplace transform expression which are highly non-linear functions of the Laplace transform variable. Thus a major challenge faced during this inversion process is the computation time required to get the transient distribution or the performance measures. It may be mentioned that it took around 180 minutes, 32 minutes and 139 minutes time to get the final output in Figures 1, 3 and 4 respectively, whereas for Figure 2 the computation time was around 2880 minutes. On comparing the parameter values for the four different figures we may find that $l = 6$ for Figures 1 and 4, $l = 20$ for Figure 2 and $l = 4$ for Figure 3. This indicates that the required computation time heavily depends on the maximum allowable size of the arriving batch (l). As in Table 1 the single arrival case was considered and the output, in the form of probabilities, was generated within a few seconds. Therefore, higher the value of l, higher will be the computation time. In fact, a small increase in l, may raise the overall computation time exponentially. To sum up, the computational expense of general queueing models (batch renewal input or general service) in transient regime is enormous, and hence it may prove to be a significant area for future research.

Further, the numerical analysis presented in this section provides valuable insights for system administrators or inventory managers handling the arrival of demands in batches of varying sizes. Understanding the inter-arrival time distribution of these batches is critical as it significantly impacts system performance. In addition, administrators should pay close attention to the transient behavior, especially during the initial stage of system operation before it stabilizes into a steady state. Given that administrators cannot directly control the arrival rate, it is advisable to prioritize increasing μ, the service rate, particularly while managing large queue lengths. This approach can help minimize potential issues and improve overall system efficiency. It is worth noting from this paper that queue length is higher for hyper-exponential distribution compared to other distributions used. The inventory manager may adjust the service rate based on the identified inter-arrival time distribution, to ensure seamless inventory management. By gaining insights from the transient behavior and the inter-arrival time distribution managers can make informed decisions to optimize system performance and resource utilization.

8. Conclusions and future work

In this paper we have performed the analysis of the infinite buffer $GI^X/M/1$ queue for its transient solution, which has not been considered before in this framework. For analytical purpose we have used the supplementary variable technique to frame the system governing equations and the difference equation method to obtain the closed-form representation of the system-content distribution in terms of its Laplace transform. Furthermore, steady-state results are deduced and compared with the existing results. The results of some of the special cases of the model are derived along with the system performance in terms of the transform variable. Further, several numerical results are presented which are obtained by inverting the non-linear Laplace transform expression using MATHEMATICA software. The results indicate that in addition to the parameter values, the choice of interarrival time distribution plays a major role in determining the system performance. Moreover, the transient behavior should also be taken into account as for certain interarrival time distribution the transition to a steady-state takes a very long time. Through the numerical examples we have also addressed the situation when the arriving batch size distribution has infinite support and the interarrival time distribution has a non-rational Laplace-Stieltjes transform.

Towards the end we have also discussed the computational complexities in terms of the required computation time, which is due to the increase in the maximum size of the arriving batch. This can be a challenging area along which future research can be carried out to make the computation of the transient distribution more feasible. Further, the model considered in this paper can be generalized to a multiserver or a vacation queueing model in the transient framework, which is left as a future work.

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