RAIRO-Oper. Res. 58 (2024) 4395–4436
https://doi.org/10.1051/ro/2024133

RAIRO Operations Research
www.rairo-ro.org

RELIABILITY DEPENDENT PRODUCTION-INVENTORY MODEL FOR REDUNDANCY ALLOCATION *VIA* FUZZY LOGIC

Anushri Maji[1], Asoke Kumar Bhunia[2] and Shyamal Kumar Mondal[1,*]

Abstract. This study deals with a reliability dependent production-inventory model in two scenarios: a redundancy allocation with crisp structure and the other with fuzzy logic. Here, a manufacturer purchases some raw-materials/components in variable cycles and arranges them as series-parallel system to produce a single item with production cost dependent on system reliability. In this model, a retailer gets the opportunity of warranty period and credit period offered by the manufacturer. Also, the retailer's demand is dependent on system reliability, credit period and selling price. In the crisp model, the component reliability is exponentially dependent on known failure rate and failure time. However, there is no dependent relationship between these two parameters. Actually, the real world is full of uncertainty and these two parameters may depend on each other following some uncertain nature which can be expressed as fuzzy logic. So, in the fuzzy model, failure time has been considered as dependent on failure rate following fuzzy logic and these fuzzy relations are defuzzified by using three fuzzy inference techniques: Mamdani, Sugeno and Tsukamoto. Main goal of this article is to determine the optimum number of cycles and components to maximize manufacturer's profit and system reliability with some constraints. The model is solved by using elitist non-dominated sorting genetic algorithm (NSGA-II) and some numerical examples closed to real-world have been executed. Comparative analyses are done for different cases; different fuzzy inference techniques and for active and mixed strategies. Finally, some sensitivity analyses and managerial insights are drawn.

Mathematics Subject Classification. 90B05, 90B25, 90B30, 90B50.

Received February 20, 2024. Accepted June 21, 2024.

Keywords. Production-reliability-inventory model, fuzzy logic, fuzzy inference techniques, RAP, NSGA-II.

[1] Department of Applied Mathematics, Vidyasagar University, Midnapore 721102, West Bengal, India.

[2] Department of Mathematics, The University of Burdwan, Burdwan 713104, West Bengal, India.

[*]Corresponding author: shyamal@mail.vidyasagar.ac.in

Abbreviations

CE-NRGA	Controlled elitism non-dominated ranked genetic algorithm
COG	Center of gravity
EPQ	Economic production quantity
FPR	Finite production rate
GA	Genetic algorithm
GHG	Green house gas
GWO	Grey wolf optimizer
LED	Light-emitting diode
MOGA	Multi-objective genetic algorithm
MOPSO	Multi-objective particle swarm optimization
NRGA	Non-dominated ranked genetic algorithm
NSGA-II	Elitist non-dominated sorting genetic algorithm
PGA	Parallel genetic algorithm
PRIM	Production-reliability-inventory model
PSO	Particle swarm optimization
RAP	Redundancy allocation problem
RRAP	Reliability redundancy allocation problem
SMO	Spider monkey optimization
VMI	Vendor managed inventory
BSC	Business set-up cost
OC	Ordering cost
RPC	Raw-materials' purchasing cost
RHC	Raw-materials' holding cost
IPC	Items' production cost
IHC	Item's holding cost
RC	Repairing cost
LI	Loss of interest

1. Introduction

System reliability plays a significant role to design different types of mechanical or electrical devices. Probably, the concept of reliability started its journey from the year 1810 and first debuted through communication and transportation systems. Till now, reliability optimization problems are applied in many real-life industrial fields such as electrical systems, mechanical systems, transformation systems, telecommunication systems, satellite systems etc. Among the several methods of prolonging system reliability, one of the best methods is to add redundant components in the system. Consequently, three types of problems are revealed such as redundancy allocation problem (RAP), reliability allocation problem and reliability redundancy allocation problem (RRAP). Many researchers work on RAP such as Mousavi *et al.* [38] worked on a multi-objective RAP with fuzzy environment by minimizing the total system cost and maximizing the system availability. Salmasnia *et al.* [44] proposed a loss function approach to solve an RAP. Zhang *et al.* [51] proposed an RRAP which is modeled with a graph. Recently, Nath and Muhuri [40] has developed an RRAP where four objectives such as system reliability, system cost, weight and volume have been optimized simultaneously providing reliability the priority most. Moreover, several systems of RRAPs (series, parallel, series-parallel, k-out-of-n, complex) include mainly three types of component configuration such as active strategy Garg and Sharma [21, 28], standby strategy and mixed strategy (active and standby components together) [5, 13, 27, 41]. In active strategy, all the components are working from the starting times and at least one component has to work any time. Standby strategy is classified into three types, cold [19, 23], warm [25, 48] and hot [39]. In cold standby strategy, one component in each subsystem is working and other components are standby and after failure of the active component, standby components will be active one-by-one by using an indicator. Here, standby components are not affected by the working components. In warm standby strategy, standby components are more affected by the active one comparing with cold standby strategy and in case of hot standby strategy, components are working like active strategy. On the other hand, in case of mixed strategy, some components are active and some are standby. Gholinezhad [22] proposed a new reliability model where redundancy strategy is a decision variable, *i.e.*, the strategy may be active or standby or mixed. **In this paper, RAP has been studied including mixed strategy with cold standby redundancy**.

In today's business conflict market situation, most of the companies prefer to reduce the total cost and maximize the total profit by using best methodologies. So, it is a practical topic for researchers to work on production-inventory model. Many researchers work on this topic. Mandal *et al.* [32] proposed an optimum production-inventory model with stock dependent demand. Mousavi *et al.* [37] developed an inventory control problem by minimizing the total cost with budget, storage space and order quantity as constraints. Betts [8] developed a hybrid simulation model of production-inventory system with cost minimization. Das *et al.* [16] maximized profit by developing a supply chain model with optimum transportation and business cycles. Das *et al.* [17] worked on an integrated production-inventory model with credit period for two types of deterioration. Manna *et al.* [33] enriched business profit by developing an economic production quantity (EPQ) model with stock dependent demand. Manna *et al.* [34] proposed an EPQ model with advertise dependent demand. Manna *et al.* [35] developed a supply chain model of two layers in which they used reliability as a parameter depending on production rate. Bhunia *et al.* [10] provided an overview of production-inventory model with flexible reliability. In their study, they took into consideration that any manufacturing factory do not have perfect production process and they may not be able to produce fully reliable products. Manna *et al.* [36] presented a production-reproduction inventory model by maximizing profit and minimizing green house gas (GHG) emission. Muhammad Al-Salamah [4] developed a production-inventory model with machine failure. A finite production rate (FPR)-based inventory model has been proposed by Chiu *et al.* [14].

1.1. Research gap and contribution

The above discussions reveal that many research works have been done on reliability optimization problem and inventory control system individually. The main goal of reliability optimization problem is to obtain the number of redundant components so that the reliability of the whole system will be maximum. But, in such type of problems, there are no discussions that how the components are maintained in the storing place. Besides, the maintenance of raw-materials and finished product is studied in the inventory control system, but it is not studied that how to increase the reliability of the finished item produced by manufacturer. Again in one side, manufacturers want most profit, on the other hand, customers prefer to buy more reliable products. Henceforth, these two systems of reliability optimization and inventory control problems may be related to each other and there should exist some research works combining these two topics. It is observed that there are very few research works on this linkage such as [2,3,43]. A single-vendor multi-retailer vendor managed inventory (VMI) model was investigated by Sadeghi *et al.* [43]. They minimized the total cost and maximized the reliability of the production system. They used redundancy in machines and ignored the part of raw-materials. So, the redundancy may be used in the raw-materials which are essential to produce items. Again, the inventory of raw-materials is also missing in their paper, it should be included. On the other hand, Alikar *et al.* [2,3] focussed on the production and transportation of the components instead of the system reliability. They showed the inventory of the components and only gave the information that the components may be used in a series-parallel system. But they were not able to address the production and inventory of the item which is produced by using the components as raw-materials. By fulfilling this critical research gap, Maji *et al.* [29] addressed a production-reliability-inventory model that incorporates a redundancy allocation problem (RAP). Their model integrates an active strategy for RAP, where the production cost is directly tied to system reliability and demand is influenced by system reliability and selling price in a linear manner. After that, Maji *et al.* [30] developed a production-reliability-inventory model (PRIM) for mixed strategy. According to Maji *et al.* [30], in their production-inventory model with reliability optimization problem, they found that demand is exponentially dependent on reliability. Additionally, they observed that demand decreases gradually with increasing selling price up to a certain threshold value, beyond which the decrease in demand follows an exponential pattern.

Moreover, they noted that the demand depends on credit period following a logistic law in their model. **In this current paper, also a PRIM has been developed for mixed strategy, with the same demand function in two scenarios: an RAP with crisp structure and the other with fuzzy logic**.

It is observed from the existing literature that the reliability of the component of any system is either known parameter or independent variable or exponentially dependent on failure rate and failure time considering these two parameters separately. Also, the failure rate may be time-dependent [24, 50, 52]. However, there is no discussion about the dependent relationship of these two parameters with uncertainty. Actually, the real-world is full of uncertainty and these two parameters depend on each other following some uncertain nature which can be expressed as fuzzy logic. By fulfilling this lacuna of the existing literature, in this paper, the relation between the failure rate and failure time has been considered by following fuzzy logic. There are many research works considering fuzzy logic in the field of inventory [11, 12, 26]. **In this paper, for the first time, fuzzy logic has been used in the PRIM as well as in the field of reliability optimization**.

In human knowledge representation, imprecision, vagueness, and approximation are common. Everyday language often uses linguistic terms like "high", "medium", and "low" to describe phenomena. Fuzzy inference aims to convert these linguistic terms into natural language expressions structured as IF-THEN type rule, *i.e.*, "IF antecedent THEN consequent". There exist three primary types of fuzzy inference method: Mamdani method, Sugeno method and Tsukamoto method. In 1975, Ebrahim Mamdani introduced the Mamdani method [31], while Takagi *et al.* [45] proposed the Sugeno method and Tsukamoto *et al.* [47] proposed the Tsukamoto method. These three methods differ in how they determine output. Mamdani's work focuses on fuzzy sets for output. In contrast, Sugeno's output membership functions are either linear or constant, whereas Tsukamoto's output are also fuzzy set but here the fuzzy sets have monotonic membership functions. The key distinction among these three methods lies in their output representation. **In our paper, we have conducted an analysis of the model, taking into account these three fuzzy inference processes**.

There are many meta-heuristic algorithms such as non-dominated ranking genetic algorithm (NRGA), controlled elitism non-dominated ranked genetic algorithm (CE-NRGA), multi-objective genetic algorithm (MOGA), non-dominated sorting genetic algorithm (NSGA-II), particle swarm optimization (PSO), grey wolf optimizer (GWO), spider monkey optimization (SMO) etc. to solve such type of complicated optimization problems which are applied in many research works [1, 9, 20, 42, 46, 49]. **In this proposed work, NSGA-II has been used which is an elitist method of genetic algorithm for solving multi-objective problems**.

The key objective of this study is to find the following queries:

- What are the optimum number of cycles and components to maximize both the manufacturer's profit and the reliability of the produced item?
- What would be the significant effect of the different cases of the model on optimum results?
- Which one of the active or mixed strategy and crisp model or fuzzy model gives better optimum decisions?
- Among the three methods of Mamdani, Sugeno, and Tsukamoto, which fuzzy inference process is the most suitable?
- How does the key parameters affect the optimum results?

To address these questions, this paper has presented a production-reliability-inventory model (PRIM) in two scenarios: one involving a redundancy allocation problem (RAP) with a crisp structure, and the other with fuzzy logic. Also, the problems have been solved by using NSGA-II with the three fuzzy inference methods of Mamdani, Sugeno and Tsukamoto. Table 1 depicts the comparison between research works related to the current study.

The rest of this paper is formulated as follows: Section 2 describes the problem and presents the notations and assumptions. Section 3 reveals the mathematical construction of crisp and fuzzy models. In Section 4, the solution methodology of NSGA-II with fuzzy inference methods has been described. Section 5 illustrates some numerical examples for the crisp and fuzzy models with sensitivity analyses. Section 6 reveals some managerial insights. Section 7 briefly discusses the overall conclusion with limitations and future research scope.

TABLE 1. Comparison between research works related to the current study.

References	Production-inventory	System reliability	Credit period	Demand	Fuzzy logic	Solution methodology
Garg and Sharma [21]	–	RRAP	–	–	–	PSO
Das *et al.* [16]	Profit	–	Quantity dependent	Constant	–	Proposed
Ardakan and Hamadani [6]	–	RRAP with mixed strategy	–	–	–	GA
Sadeghi *et al.* [43]	Cost	RAP in series-parallel system	–	Deterministic and known	–	NSGA-II
Das *et al.* [17]	Profit	–	Fuzzy	Constant	–	Proposed
Manna *et al.* [34]	Profit	–	–	Advertise dependent	–	Proposed
Kim and Kim [27]	–	RRAP with active or cold redundancy	–	–	–	PGA
Alikar *et al.* [3]	Cost	RAP with active strategy	–	Constant	–	CE–NRGA, NSGA-II, MOPSO
Banu and Mondal [7]	Profit	–	Warranty period dependent	Warranty and credit period dependent	–	Proposed
Ouyang *et al.* [41]	–	RRAP with mixed strategy	–	–	–	PSO
Chakraborty *et al.* [12]	Profit	–	–	Selling price dependent	Demand and selling price	GA
Maji *et al.* [29]	Profit	RAP with active strategy	–	Reliability and selling price	–	MOGA
Maji *et al.* [30]	Profit	RAP with mixed strategy compared with active strategy	Warranty period dependent	Reliability-selling price-credit period dependent	–	NSGA-II
Current paper	**Profit**	**Crisp and fuzzy RAPs with mixed strategy compared with active strategy**	**Warranty period dependent**	**Reliability-selling price-credit period dependent**	**Failure time and failure rate**	**NSGA-II**

2. Problem description

In this study, a reliability dependent production-inventory model (Figure 1), called a production-reliability-inventory model (PRIM) [29,30], has been developed for a profit maximization problem along with a reliability optimization problem with some resource constraints. This model deals with a single manufacturer and a single retailer. Here, manufacturer produces a single type of item whose configuration has n parts, known as subsystem which are connected serially to each other to give the service. If any part is failed to work then item *i.e.*, system stops its functioning. So, to increase the reliability of the item, here it is considered that each part is constituted with same type of several components which are connected in parallel and the system becomes a series-parallel system. So, to produce an item, the manufacturer purchases different n types of raw-materials instantaneously from suppliers in some cycles (η) of equal time length (T_0) and stores them in a warehouse. In each part *i.e.*, in each subsystem, the raw-materials *i.e.*, the components are arranged using mixed strategy which indicates that some of the components are active (x_i number of components for i^{th} subsystem, $i = 1, 2, \ldots, n$) and some are cold standby (y_i number of components for i^{th} subsystem). In case of a single subsystem, after failure of last active component, the component is replaced with the first standby component. There are two types of indicator/switch to detect the failure of components [15]. In this current model, constant switching reliability ($\rho = 0.99$) has been used for each subsystem. In this way, the items are produced and also they carry some volume, cost, weight which are dependent on the respective volume, cost, weight of the components. The manufacturer sells finished products to the retailer according to retailer's demand. To increase retailer's demand, the manufacturer allows some relaxation in payment period (*i.e.*, credit period) and also offers warranty period of the product. For offering credit period, the manufacturer has to deal with a loss of interest. In case of warranty period, if the failure time (t) of the product is less than or equal to the warranty period (W) then the manufacturer has to bear the repairing cost and when the failure time is greater than the warranty period then the manufacturer repairs the product with earning of repairing cost from the retailer.

In the crisp model, the component reliability exponentially depends on the failure rate of the component and the failure time of the system. Here, the failure rate and the failure time are known and considered as separate parameters. In the fuzzy model, the component reliability is also dependent on the failure rate and the failure time, but here, failure time is depending on the failure rate following fuzzy logic. Here, the value of the failure time has been derived from three well-known fuzzy inference techniques of Mamdani, Sugeno and Tsukamoto and treated as parameter. So, the problem becomes related to the RAP. Consequently, the main objective of the crisp model as well as fuzzy model is to determine the optimum number of cycles and components to maximize manufacturer's profit as well as reliability of the system/item with some constraints of volume, cost and weight.

2.1. Notations

To developed the proposed model, following notations have been used.

- n: Number of subsystems in the series-parallel configuration as well as the type of components
- x: $(x_1, x_2, \ldots, x_n)$, where x_i is the number of active redundant components used in the i^{th} subsystem (decision variable)
- y: $(y_1, y_2, \ldots, y_n)$, where y_i is the number of cold standby components used in the i^{th} subsystem (decision variable)
- t_i: Failure time of the component used in the i^{th} subsystem
- t: Failure time of the system
- λ_i: Failure rate of the component used in the i^{th} subsystem
- ρ: Constant failure indicator/switching reliability at time t
- r: $(r_1, r_2, \ldots, r_n)$, where r_i is the reliability of each component of i^{th} subsystem (dependent on failure rate and failure time)
- $R(t; x, y)$: System reliability at time t dependent on number of active and cold standby components
- $f_{i,x_i}(t)$: Probability density function of the time t when all the active components of i^{th} subsystem fail
- $f_{i,l}(t)$: Probability density function of the time t when l^{th} cold standby component of i^{th} subsystem fail

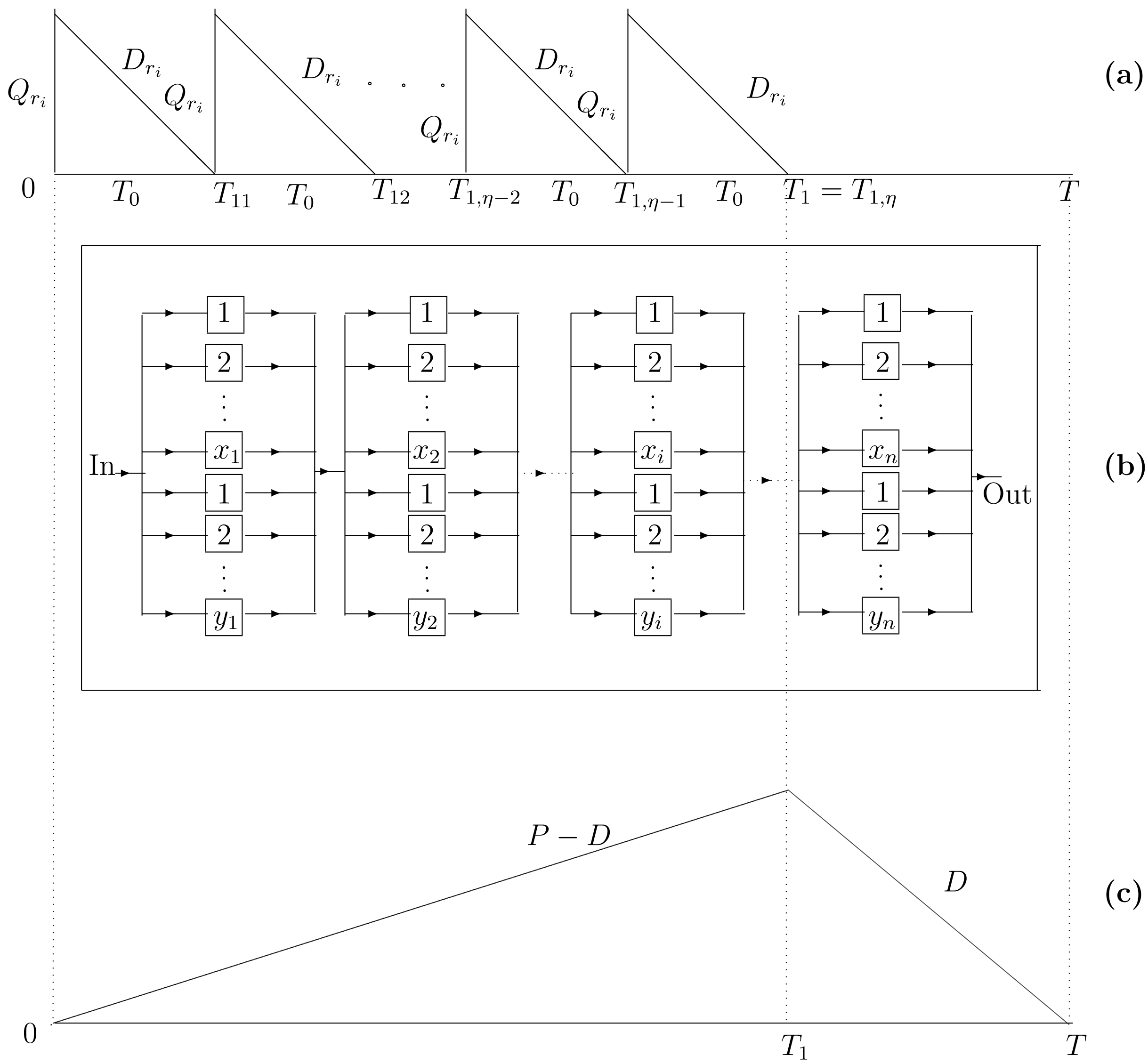


FIGURE 1. Reliability dependent production-inventory model.

– D_{r_i}: Demand rate of the raw-materials used in i^{th} subsystem
– D: Demand rate of the produced item
– P: Production rate of the item
– Q_{r_i}: Lot-size of raw-materials used in the i^{th} subsystem
– T_0: Time duration of each cycle
– η: The number of cycles in the raw-material's stocking period (decision variable)
– c_1: Set-up cost of the business
– c_2: Ordering cost of raw-materials per cycle
– c_{3_i}: Purchasing cost of per unit raw-materials used in the i^{th} subsystem
– c_{4_i}: Holding cost of raw-materials used in the i^{th} subsystem per unit per unit time

– c_p: Production cost of finished product per unit item
– c_5: Holding cost of finished product per unit per unit time
– s: Selling price of finished product per unit
– W: Warranty period of the finished product
– c_r: Repairing cost per unit item
– θ: Rate of damaged item
– ξ: Rate of damaged item which is returned to the manufacturer for repairing
– M: Credit period offered by manufacturer to the retailer
– I_p: Rate of interest which manufacturer loss due to offering credit period
– T_1: Total time horizon of the stocking of raw-materials *i.e.*, total time of production of items
– T: Total time of business
– Z: Total profit of manufacturer

2.2. Assumptions

Following assumptions are formulated to build the model.

(i) A manufacturer purchases n types of raw-materials from suppliers instantaneously in different cycles (η) of equal time length T_0, $T_0 = \dfrac{T_1}{\eta}$, where T_1 is raw-materials' stocking period.
(ii) Using these n types of raw-materials as components in n subsystems of a series-parallel system, the manufacturer produces items of single type at the rate of P and sells them to a retailer according to the retailer's demand (D), obviously $P > D$.
(iii) The series-parallel system with a mixed strategy consists of n subsystems which contain $x_i (i = 1, 2, \ldots, n)$ active and $y_i (i = 1, 2, \ldots, n)$ cold standby components in i^{th} subsystem. The components are identical within a particular subsystem and differ from subsystem to subsystem, *i.e.*, n type of raw-materials (components) are used in n number of subsystems to produce a single item (system).
(iv) The system is in mixed strategy with constant switching reliability ($\rho = 0.99$) [30].
(v) The components have three different states: operating, standby and failure state.
(vi) For the i^{th} component, the component reliability (r_i) is represented as exponential functions of failure rate (λ_i) and failure time (t) such as $r_i = e^{-\lambda_i t}$, for $i = 1, 2, \ldots, n$ [27,30], where failure time of a system means the time after which the system fails. In the crisp model, the failure rate and the failure time have been treated as distinct and known parameters. Despite the dependency of component reliability on both the failure rate and failure time, there is a notable absence of discussion in the literature regarding the uncertain relationship between these two parameters. However, real-world scenarios exhibit a correlation between failure time and failure rate, reflecting the uncertainties. Rather than precise values, linguistic terms are commonly used to describe this relationship. For instance, systems with high failure rates are characterized by quick failures, resulting in shorter failure times. Conversely, systems with low failure rates tend to enjoy high failure times, while those with medium failure rates fall in medium failure time. Hence, a discernible relationship exists between failure time and failure rate, articulated through linguistic terms such as "Low","Medium", "High" etc. With these considerations in mind, the fuzzy model has introduced a scenario where failure rate and failure time are represented as linguistic terms instead of fixed parameters. Unlike the crisp model, the fuzzy model accommodates the failure times (t_i) based on the failure rate (λ_i) of the i^{th} component, employing IF-THEN rule-based fuzzy logic. Here, the rules have been considered as follows:

Rule 1: IF (Failure rate λ_i is Low) THEN (Failure time t_i is High).
Rule 2: IF (Failure rate λ_i is Medium) THEN (Failure time t_i is Medium).
Rule 3: IF (Failure rate λ_i is High) THEN (Failure time t_i is Low).

where the linguistic terms "Low", "Medium", and "High" for the failure rate input are depicted as triangular fuzzy numbers and for the failure time output, they are different for different fuzzy inference techniques.

(vii) The cost of each component of the i^{th} subsystem is defined as

$$c(r_i) = \alpha_i \Big(\frac{-t}{log r_i}\Big)$$

where α_i is the positive parameter [6,30].

(viii) To increase the demand of produced item, the manufacturer offers a credit period (M) to the retailer which is dependent on warranty period (W) of item such that

$$M = M_0 - \delta W$$

where M_0 is the initial credit period when there is no warranty period of the item and δ is the effective parameter of warranty period on credit period [7,30].

(ix) Similar as Maji *et al.* [30], the demand function is dependent on system reliability, selling price and credit period following the equation:

$$D = \begin{cases} D_0 e^{\alpha(R-R_0)} - D_1(s-s_0)^q + \dfrac{D_3}{1+e^{-k(M-M_0)}} & ;\text{if } s_0 \leq s \leq s_1 \\ D_0 e^{\alpha(R-R_0)} - D_2 e^{\beta(s-s_1)} + \dfrac{D_3}{1+e^{-k(M-M_0)}} & ;\text{if } s \geq s_1 \end{cases}$$

(x) Manufacturer has to face a loss of interest (I_c) due to offer credit period which is called opportunity cost.

(xi) Repairing policy has been considered similar as Maji *et al.* [30]. When any item is damaged within the warranty period ($t \leq W$) then the manufacturer has to repair it in free of cost and when any item damaged after the warranty period ($t > W$), then manufacturer earns the repairing cost from the retailer.

(xii) Production cost per unit item (c_p) is dependent on system reliability following the equation:

$$c_p = c_{p_0} + c_{p_1}(R - R_0)$$

where c_{p_0} is the initial production cost of per unit item when the products have the reliability R_0, c_{p_1} is effective parameter and R_0 is the initial system reliability [29,30].

(xiii) Shortages are not allowed and raw-materials are available in instant time.

(xiv) Lead time is negligible.

3. Mathematical construction of the model

In this section, we have discussed the mathematical construction of the model in two scenarios such as (1) crisp model and (2) fuzzy model.

3.1. Crisp model

The main objective of the crisp model is to determine the optimum number of cycles and components to maximize manufacturer's profit as well as reliability of the system/item with some constraints of volume, cost and weight. Here, the component reliability exponentially depends on known failure rate and failure time.

The manufacturer's profit is entrusted with business set up cost, ordering, purchasing and holding cost of raw-materials, production and holding cost of finished item, repairing cost, loss of interest and sales revenue.

(i) **Business set-up cost:** Total business set-up cost, $BSC = c_1$.

(ii) **Raw-materials' ordering cost:** Total ordering cost, $OC = \eta c_2$.

(iii) **Raw-materials' purchasing cost:** In each cycle, manufacturer purchases Q_{r_i} quantity of raw-materials of i^{th} type. So, raw-materials' purchasing cost,
$RPC = \sum_{i=1}^{n} c_{3_i} \eta Q_{r_i}$

(iv) **Raw-materials' holding cost:** Raw-materials' holding cost, RHC is given by
$$RHC = \sum_{i=1}^{n} \eta c_{4_i} \Big[-\frac{1}{2} D_{r_i} T_0^2 + Q_{r_i} T_0 \Big] = \sum_{i=1}^{n} \frac{1}{2} \eta c_{4_i} P(x_i + y_i) T_0^2$$
[Detailed calculation is given in Appendix A.]

(v) **Item's production cost:** Items' production cost, $IPC = PT_1 c_p$.

(vi) **Item's holding cost:** Items' holding cost, IHC is given by
$$IHC = \frac{1}{2} c_5 \frac{DT^2}{P} \Big(P - D \Big)$$
[Detailed calculation is given in Appendix B.]

(vii) **Repairing cost(RC):** Here, two types of repairing costs are available according to assumption (xi). Within the warranty period ($t \leq W$), the manufacturer has to bear the repairing cost RC_1, where

$$RC_1 = c_r \theta W D$$

Again, after the warranty period ($t > W$), the manufacturer earns the repairing cost RC_2 from the retailer which is given by

$$RC_2 = c_r \theta (T_1 - W) D\xi = c_r \theta \Big(\frac{DT}{P} - W \Big) D\xi$$

(viii) **Loss of interest:** For the credit period, manufacturer has to deal with a loss of interest, $LI = I_p M s D$.

(ix) **Sales revenue:** Manufacturer's sales revenue is as follows:
$$Rev = s \int_0^T D \, dt = sDT$$

Now, the total profit (Z) of the manufacturer is given by

$$Z = \text{Rev -}\Big(\text{BSC + OC + RPC + RHC + IPC + IHC + RC + LI}\Big)$$

$$\begin{aligned} i.e., Z(x, y, \eta) = sDT - \Bigg(& c_1 + \eta c_2 + \sum_{i=1}^{n} c_{3_i} DT(x_i + y_i) + \sum_{i=1}^{n} \frac{1}{2} c_{4_i} (x_i + y_i) \frac{(DT)^2}{P\eta} + DTc_p \\ & + \frac{1}{2} c_5 \frac{DT^2}{P} (P - D) + RC + I_p M s D \Bigg) \end{aligned}$$

where, $RPC = \sum_{i=1}^{n} c_{3_i} \eta Q_{r_i} = \sum_{i=1}^{n} c_{3_i} DT(x_i + y_i)$, $\quad IPC = PT_1 c_p = DTc_p$ and
$$RC = \begin{cases} RC_1 & ; \text{when } t \leq W \\ -RC_2 & ; \text{when } t > W \end{cases}$$
Here, negative sign indicates the earning of repairing cost of the manufacturer from the retailer. So, RC_1 is the payable cost and RC_2 is the income of the manufacturer.

System reliability related formulation

The system of the produced item contains n subsystems in series and the components in each subsystem are in parallel configuration. Each subsystem contains a mixed type of redundancy, *i.e.*, there are active and cold standby components both Figure 1(b). For the mathematical calculation of system reliability formulation one may go through the work of Maji *et al.* [30].

The system reliability with mixed strategy is given by

$$R(t;x,y) = \prod_{i=1}^{n}\Bigg[\Big\{1-(1-r_i)^{x_i}\Big\} + \rho\int_0^t f_{i,x_i}(t_1) r_i(t-t_1)\,dt_1 + \sum_{l=1}^{y_i-1}\rho^{l+1}\int_0^t\int_{t_1}^t f_{i,x_i}(t_1) f_{i,l}(u) r_i(t-u)\,du\,dt_1\Bigg] \tag{1}$$

Here, subsystems reliability are sum of some mutually exclusive events. The first term, $\{1-(1-r_i)^{x_i}\}$ shows the probability that all active components are working and there is no needs of cold standby components. The second term indicates the probability when all active components fail at time t_1 and first cold standby component starts to work until the system's failure time t. The third term shows the sum of probabilities when l^{th} cold standby component fails and $(l+1)^{th}$ cold standby component works until the system's failure time t.

Here, failure time is exponentially distributed. Probability density function (PDF), $f_i(t) = \lambda_i e^{-\lambda_i t}$ and cumulative distribution function (CDF), $F_i(t) = 1 - e^{-\lambda_i t}$. So, PDF of the time when all the active components fail, follows the condition (2).

$$f_{i,x_i}(t) = x_i \times [F_i(t)]^{x_i-1} \times f_i(t). \tag{2}$$

Equation (2) indicates that there were x_i number of active components which have been failed. After that, only one component (l^{th} component, $l = 1, 2, \ldots, y_i - 1$) will be active one by one among the all y_i number of standby components. So, the PDF of the time when l^{th} component fails is given by

$$f_{i,l}(t) = f_i(t)$$

($\because$ there is only one active component).

Lemma 1. $f_{i,x_i}(t) = x_i \times [F_i(t)]^{x_i-1} \times f_i(t)$.

Proof. Probability density function (PDF), $f_i(t) = \lambda_i e^{-\lambda_i t}$ and cumulative distribution function (CDF), $F_i(t) = 1 - e^{-\lambda_i t}$.

So, $\int_0^t f_{i,x_i}(t_1)\,dt_1$
= CDF of time when all x_i components fail
= 1 - (reliability of system containing x_i components in the i^{th} subsystem)
$= (1-e^{-\lambda_i t})^{x_i}$
$= \Big[F_i(t)\Big]^{x_i}$

After differentiation we get,

$f_{i,x_i}(t) = x_i \times [F_i(t)]^{x_i-1} \times \dfrac{d}{dt}[F_i(t)]$
$\therefore f_{i,x_i}(t) = x_i \times [F_i(t)]^{x_i-1} \times f_i(t)$

Hence the proof. □

Now, equation (1) becomes as

$$R(t;x,y)=\prod_{i=1}^{n}\Bigg[1-(1-e^{-\lambda_i t})^{x_i}+\rho x_i\lambda_i t e^{-\lambda_i t}-\Big(\rho x_i+\frac{\rho^2-\rho^{y_i+1}}{1-\rho}\Big)e^{-\lambda_i t}\sum_{p=1}^{x_i-1}\frac{(1-e^{-\lambda_i t})^p}{p}+\Big(\frac{\rho^2-\rho^{y_i+1}}{1-\rho}\Big)e^{-\lambda_i t}\Big\{\lambda_i t-\frac{(1-e^{-\lambda_i t})^{x_i}}{x_i}\Big\}\Bigg]$$

[Detailed calculation is given in Appendix C.]

So, for mixed strategy, the multi-objective non-linear constrained optimization problem of crisp model is given by following equation (3)

$$\left.\begin{aligned}
&\text{maximize}\;\; Z(x,y,\eta)=sDT-\Bigg(c_1+\eta c_2+\sum_{i=1}^{n}c_{3_i}DT(x_i+y_i)\\
&\qquad+\sum_{i=1}^{n}\tfrac{1}{2}c_{4_i}(x_i+y_i)\tfrac{(DT)^2}{P\eta}+DTc_p+\tfrac{1}{2}c_5\tfrac{DT^2}{P}(P-D)\\
&\qquad+RC+I_pMsD\Bigg)\\
&\text{maximize}\;\; R(t;x,y)=\prod_{i=1}^{n}\Bigg[1-(1-e^{-\lambda_i t})^{x_i}\\
&\qquad+\rho x_i\lambda_i t e^{-\lambda_i t}-\Big(\rho x_i+\frac{\rho^2-\rho^{y_i+1}}{1-\rho}\Big)e^{-\lambda_i t}\sum_{p=1}^{x_i-1}\frac{(1-e^{-\lambda_i t})^p}{p}\\
&\qquad+\Big(\frac{\rho^2-\rho^{y_i+1}}{1-\rho}\Big)e^{-\lambda_i t}\Big\{\lambda_i t-\frac{(1-e^{-\lambda_i t})^{x_i}}{x_i}\Big\}\Bigg]\\
&\text{subject to}\\
&\sum_{i=1}^{n}v_i(x_i+y_i)\le V,\;\;\sum_{i=1}^{n}c(r_i)(x_i+y_i)\le C,\;\;\sum_{i=1}^{n}w_i(x_i+y_i)\le W_1
\end{aligned}\right\}\quad(3)$$

For active strategy, the multi-objective non-linear constrained optimization problem for crisp model is given by following equation (4)

$$\left.\begin{aligned}
&\text{maximize}\;\; Z(x,\eta)=sDT-\Bigg(c_1+\eta c_2+\sum_{i=1}^{n}c_{3_i}DTx_i+\sum_{i=1}^{n}\frac{1}{2}c_{4_i}x_i\frac{(DT)^2}{P\eta}\\
&\qquad+DTc_p+\tfrac{1}{2}c_5\tfrac{DT^2}{P}(P-D)+RC+I_pMsD\Bigg)\\
&\text{maximize}\;\; R(t;x)=\prod_{i=1}^{n}\Big[1-(1-e^{-\lambda_i t})^{x_i}\Big]\\
&\text{subject to}\\
&\sum_{i=1}^{n}v_ix_i\le V,\;\;\sum_{i=1}^{n}c(r_i)x_i\le C,\;\;\sum_{i=1}^{n}w_ix_i\le W_1
\end{aligned}\right\}\quad(4)$$

3.2. Fuzzy model

In the fuzzy model, the component reliability (r_i) of the system has been considered as dependent on the failure rate (λ_i) and failure time (t_i) for the i^{th} component following the equation $r_i=e^{-\lambda_i t_i}$, for $i=1,2,\ldots,n$. Again, according to assumption (vi), the failure time (t_i) depends on the failure rate (λ_i) of the i^{th} component using IF-THEN rule-based fuzzy logic where the failure time of the whole system (t) has been calculated as the minimum of the failure time of the components, *i.e.*,

$$t = \min\{t_i\}, \text{ for } i = 1, 2, \ldots, n. \tag{5}$$

The reason behind this can be explained in this way: Here, t_i is the failure time of the i^{th} component. The components are identical within each i^{th} subsystem but vary from subsystem to subsystem (Figure 1(b)). The whole system fails when any one of the subsystem fails because the subsystems are connected in series configuration. Hence, the time after which the whole system fails, t, is the minimum time of the failure of any of the i^{th} subsystem *i.e.*, t_i.

The value of this failure time has been derived from three popular fuzzy inference techniques such as Mamdani, Sugeno and Tsukamoto methods (details about these fuzzy inference techniques are described in Appendix D). The primary aim of this fuzzy model is to determine the optimum number of cycles of collecting raw-materials and the optimum number of components to maximize both manufacturer's profit and the reliability of the whole system or item, while adhering to constraints related to volume, cost, and weight.

4. Solution methodology

In the crisp as well as fuzzy models of this study, there are integer-type decision variables x_i and η for the active strategy, and x_i, y_i, and η for the mixed strategy, where $i = 1, 2, \ldots, n$. In the crisp model, the component reliability (r_i) is a function of λ_i and t where these two parameters are independent. On the other hand, in the fuzzy model, r_i is a function of λ_i and t_i where t_i depends on λ_i following fuzzy logic. In this paper, all the problems are non-linear constrained multi-objective optimization problems. Consequently, these problems cannot be solved analytically. An elitist non-dominated sorting genetic algorithm (NSGA-II) [18] has been utilized among the available methods for addressing these multi-objective problems. In dealing with the crisp model problems, NSGA-II has been directly applied. In contrast, for the fuzzy model problems, first the value of the failure time t_i has been determined using three well-known fuzzy inference techniques: the Mamdani method, the Sugeno method, and the Tsukamoto method. After that, the value of t is obtained by using equation (5). Subsequently, NSGA-II has been applied, treating the obtained value of t as input in the NSGA-II process. The detailed procedure is given as follows:

For better understanding of NSGA-II, one may go through the work of Deb [18]. Here, we have discussed about the above-mentioned fuzzy inference techniques.

(1) Mamdani method:

According to the three rules of fuzzy logic described in assumption (vi), the failure rate input employs three distinct membership functions of triangular fuzzy numbers associated with the linguistic terms "Low" (L), "Medium" (Med), and "High" (H), within the domain of (0,1). While the structure of the input remains consistent, the output (failure time) varies for different fuzzy inference techniques. In the case of the Mamdani method, the output is also defined with membership functions of triangular fuzzy numbers corresponding to "Low" (L), "Medium" (Med), and "High" (H), but within a different domain of (0,3). Figure 2 displays the input, output, and corresponding membership functions for Mamdani method.

- **Fuzzification of input λ_i:** After selecting the domain of λ_i, Figure 3 shows that the input value $\lambda_1 = 0.22$ corresponds to two fuzzy numbers of "Low" and "Medium" with the membership values 0.9 and 0.1 respectively.
- **Rule evaluation and rule strength calculation:** Evaluating the rules described in assumption (vi), Figure 4 reveals that the corresponding membership value which is known as rule strength, is 0.9 for the Rule 1 and 0.1 for the Rule 2. Also, Rule 3 is not under consideration as λ_1 does not correspond to the fuzzy number "High".

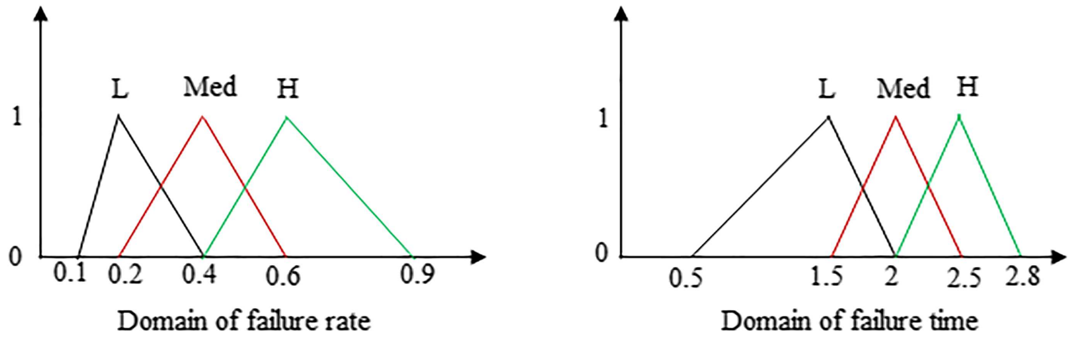


FIGURE 2. Failure rate input and failure time output with their degree of membership for Mamdani method.

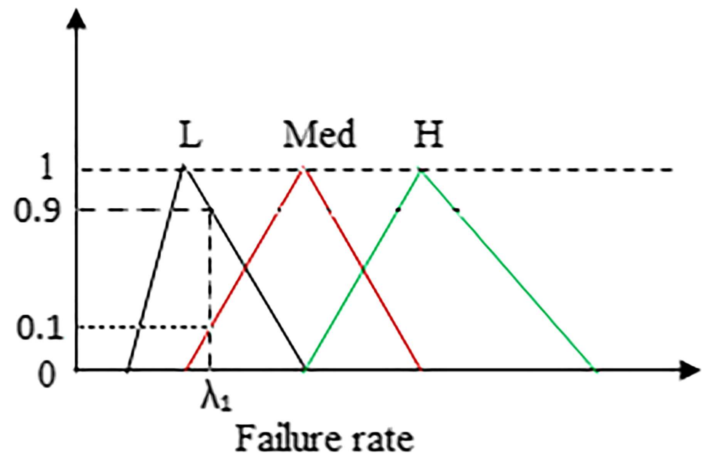


FIGURE 3. Fuzzification of the failure rate input for Mamdani method.

– **Fuzzy output:** After calculating the rule strength for each rule, the region bounded by the rule strength line is known as fuzzy output which is depicted in Figure 5.
– **Defuzzification:** To derive the ultimate crisp value for the output t_1, the centroid method has been applied within the enclosed region.

In this way, all the values of individual $t_i, i = 1, 2, \dots, n$ has been calculated and the value of t has been calculated by using equation (5).

(2) Sugeno method:

In this method, the domain of the antecedent remains same as the Mamdani method, only difference in the consequent part. Here, the consequent part is a function of the antecedent. In this method, the failure time (t_i) depends on the failure rate (λ_i) under the following fuzzy rules:

$$\left.\begin{array}{l}\textbf{Rule 1: } \text{IF } \lambda_i \text{ is Low THEN } t_i \text{ is } t_i = -a_1\lambda_i + b_1 \\ \textbf{Rule 2: } \text{IF } \lambda_i \text{ is Medium THEN } t_i \text{ is } t_i = -a_2\lambda_i + b_2 \\ \textbf{Rule 3: } \text{IF } \lambda_i \text{ is High THEN } t_i \text{ is } t_i = -a_3\lambda_i + b_3\end{array}\right\} \tag{6}$$

where a_1, a_2, a_3, b_1, b_2 and b_3 are positive scalars.

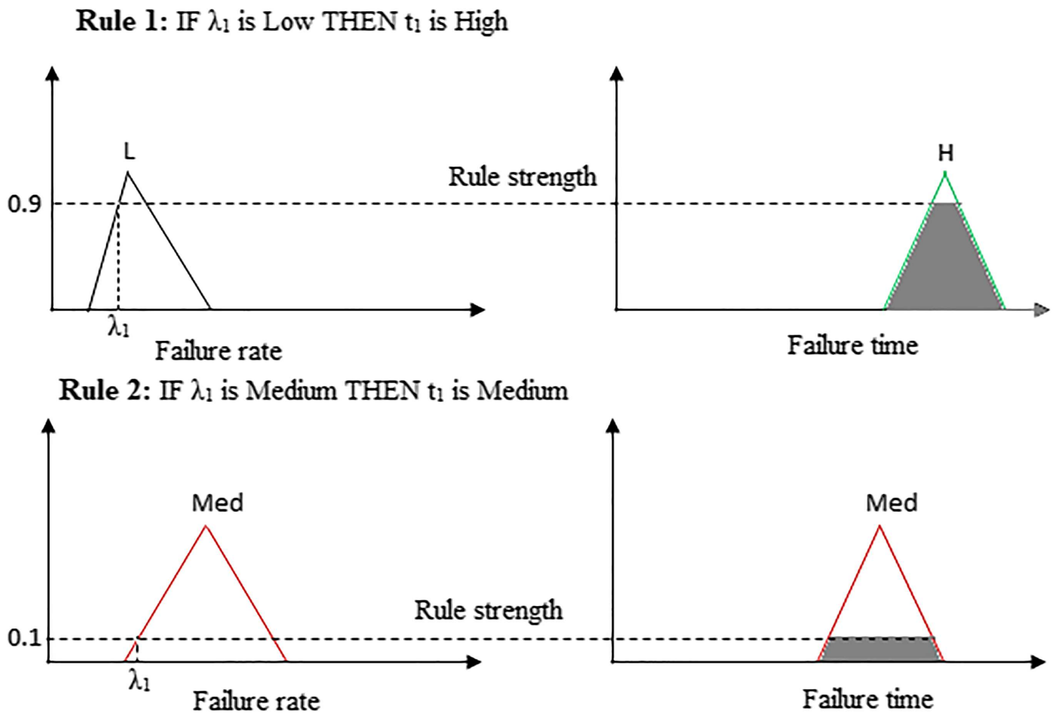


FIGURE 4. Rule evaluation and rule strength calculation for Mamdani method.

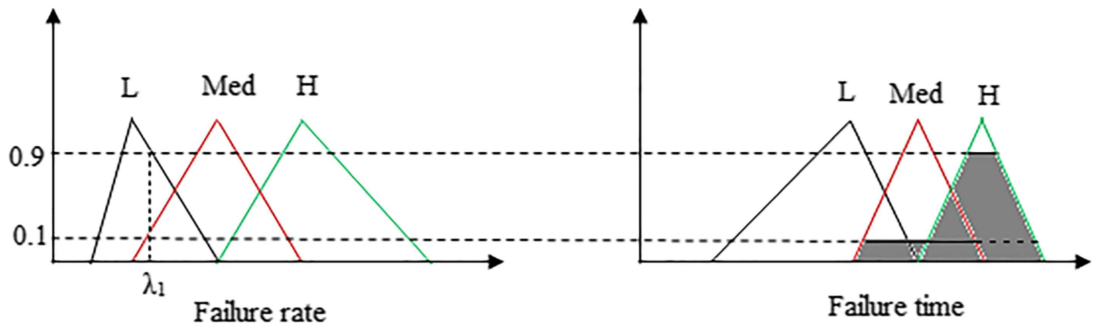


FIGURE 5. Fuzzy output for Mamdani method.

Here, rule strength for each rule is calculated similar as the Mamdani method. So, corresponding to $\lambda_1 = 0.22$, the rule strength for Rule 1 is 0.9, for Rule 2 is 0.1 and for Rule 3 is 0. Figure 3 supports the fact. Also, the value of t_i for each rule which is denoted by $t_i^j (j = 1, 2, 3)$, is calculated from the rules described in equation (6). The final crisp output t_i is obtained by applying the weighted-average mechanism,

$$t_i = \frac{\sum_{j=1}^{3} w_i^j t_i^j}{\sum_{j=1}^{3} w_i^j}$$

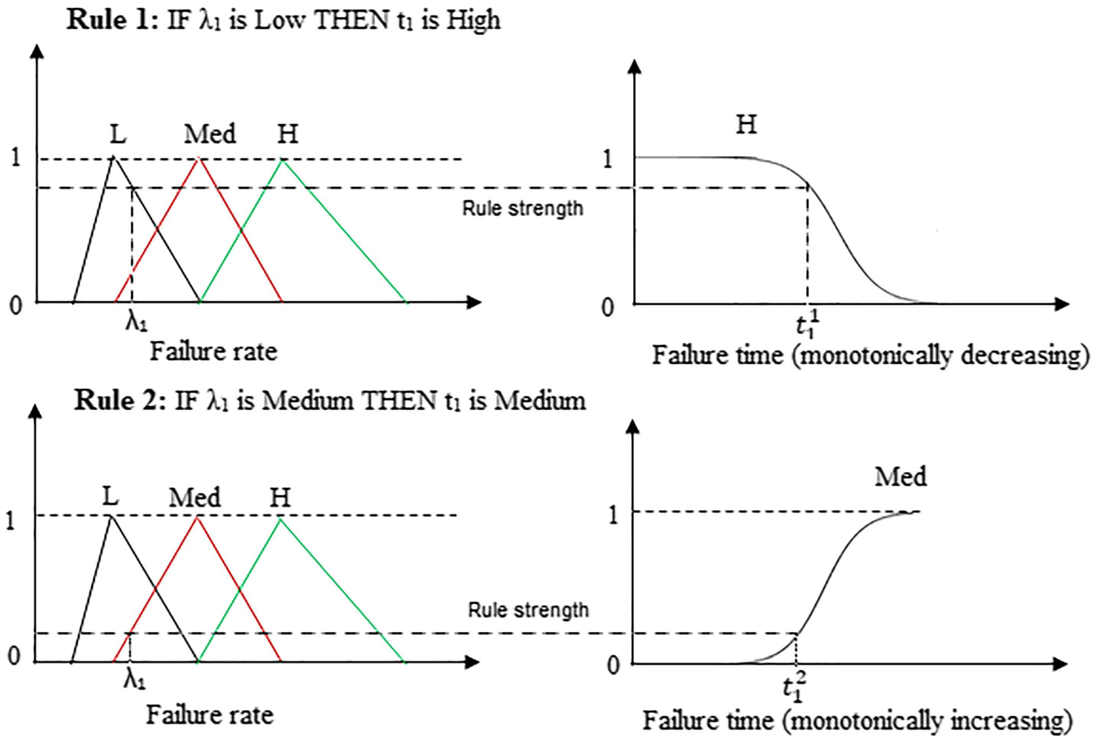


FIGURE 6. Failure rate input and failure time output with their degree of membership for Tsukamoto method.

where $w_i^j (j = 1, 2, 3)$ is the rule strength for j^{th} rule. After that, the value of t has been calculated by using equation (5).

(3) Tsukamoto method:

In the Tsukamoto method, both the antecedent and consequent of the three rules are described as fuzzy numbers like Mamdani method. But here, the consequent part has monotonic membership function. So, we have considered the antecedent part as triangular fuzzy numbers same as the Mamdani method and the consequent part as the monotonic function of logistic curve described in Figure 6 where the membership functions of failure time t_i: "High", "Medium" and "Low" are the monotonic function of logistic curve. This logistic curve can be defined as

$$w_i^j = \frac{1}{1 + e^{-k_1(t_i^j - t_0)}}$$

where k_1 is the logistic growth rate, t_0 is the sigmoid midpoint, w_i^j is the rule strength and t_i^j is the corresponding value of failure time for $j^{th}(j = 1, 2, 3)$ rule. When k_1 is negative, the logistic curve becomes a monotonically decreasing function and when k_1 is positive, it becomes a monotonically increasing function. Also in Figure 6, Rule 3 is not under consideration for λ_1 as λ_1 does not correspond to the fuzzy number "High".

Here also, the rule strength is calculated similar as previous two methods and the value of t_i^j is calculated from the corresponding rule strength of the j^{th} rule. Consequently, the final crisp value of the output t_i is obtained by using weighted-average method similar as the Sugeno method and the value of t has been calculated by using equation (5).

The outline of the entire procedure of obtaining optimum solution is described in the following algorithm:

Algorithm for optimum solution

Step 1: Input all the relevant parameters associated to the model.
Step 2: Set the domain of the variables x_1, x_2, y_1, y_2 and η.
Step 3: For getting optimum solution of the crisp model input the failure time t as parameter. For the fuzzy model go to the next step.
Step 4: Set $i = 1$.
Step 5: Set the domain of failure rate λ_i and failure time t_i (Figure 2).
Step 6: Input the value of λ_i.
Step 7: Determine the corresponding membership values of λ_i for the Low, Medium and High fuzzy sets (Figure 3).
Step 8: Evaluate the rules for Mamdani, Sugeno and Tsukamoto methods.
Step 9: In case of the Mamdani method (Figure 4) and Tsukamoto method (Figure 6), for each rule, the respective membership values of λ_i for Low, Medium and High correspond to the membership function High, Medium and Low of the output t_i. Obtain these membership values as rule strength denoted by w_i^j(here, $j = 1, 2, 3$) for j^{th} rule. For the Sugeno method, obtain the respective membership values of λ_i as rule strength for each rule.
Step 10: For Mamdani method, set the degree of membership of the output t_i according to the rule strength for the various output triangular fuzzy numbers Low, Medium and High. For Sugeno method, find the value of t_i for each rule from equation (6) where t_i^j is the value of t_i in the j^{th} rule (here, $j = 1, 2, 3$) and for Tsukamoto method, find the value of t_i for each rule from the corresponding rule strength for the output monotonic fuzzy numbers (Figure 6).
Step 11: For Mamdani method, find the region bounded by the rule strength line for each rule (Figure 4). For the Sugeno and Tsukamoto methods go to Step 14.
Step 12: For Mamdani method, determine the fuzzy output (Figure 5) applying aggregation operator on the region obtained in Step 11.
Step 13: Apply the centroid method on the fuzzy output to obtain the defuzzified failure time (t_i) with the formula:

$$t_i = \frac{\sum_{u=1}^{m} A_u c_u}{\sum_{u=1}^{m} A_u}$$

where A_u is the area and c_u is the centroid of the u^{th} sub-region.
Step 14: For Sugeno and Tsukamoto methods, find the final crisp output t_i by applying the weighted-average mechanism,

$$t_i = \frac{\sum_{j=1}^{3} w_i^j t_i^j}{\sum_{j=1}^{3} w_i^j}$$

where w_i^j is the rule strength and t_i^j is the output value of t_i for j^{th} rule, $j = 1, 2, 3$ here.
Step 15: Set $i = i + 1$ and repeat the steps from Step 4 to Step 14 upto $i = n$.
Step 16: Obtain the failure time of the whole system (t) by applying the formula t=min$\{t_i\}, i = 1, 2, \ldots, n$.
Step 17: Insert this value of t as input for the fuzzy model in the process of NSGA-II.
Step 18: For both the crisp and fuzzy models, apply NSGA-II to obtain the final non dominated set of solutions.

Table 2. Data for series-parallel system.

n	c_{3_i}	c_{4_i}	λ_i	α_i	v_i	w_i	V	C	W_1	t	ρ
1	0.75	0.08	0.22	0.165	0.9	5	60	50	350	0.75	0.99
2	0.5	0.05	0.25	0.125	0.8	6					

Table 3. Data for business

M_0	δ	W	D_0	D_1	D_3	α	R_0	s	s_0	q	k	c_{p_0}	c_{p_1}	T	c_1	c_2	P	c_5	c_r	θ	I_p
0.083	0.01	1	1500	3	3000	3	0.5	250	220	0.8	0.1	100	200	3	300000	500	8500	0.5	20	0.05	2

5. Numerical analysis

In this section, previously mentioned crisp and fuzzy models have been analyzed for active strategy as well as mixed strategy with the help of some numerical examples related to the production of LED lamp.

5.1. Results for crisp model

In this subsection, four numerical examples have been illustrated for the crisp model described in Subsection 3.1 according to four different cases based on failure time (t), warranty period (W) and selling price (s) such as: Case-I (When $t \leq W$, $s_0 \leq s \leq s_1$), Case-II (When $t \leq W$, $s \geq s_1$), Case-III (When $t > W$, $s_0 \leq s \leq s_1$) and Case-IV (When $t > W$, $s \geq s_1$). On the basis of these cases, comparisons have been done between active and mixed strategies. All the examples of the crisp model have been solved by applying NSGA-II directly.

Example 1 (When $t \leq W$, $s_0 \leq s \leq s_1$):

Here, a numerical example has been evaluated which is related to Case-I of the crisp model when failure time (t) does not exceed the warranty period (W) and the selling price (s) lies in a certain value (between s_0 and s_1). The example is as follows:

In variable cycle (η) with equal time length (T_0), a manufacturer purchases two types of small LED bulbs as raw-materials for the production of LED lamps at the rate of 8500 unit per year at prices \$0.75 and \$0.50 respectively and holds them in a warehouse at cost of \$0.08 and \$0.05 respectively per unit raw-materials per year. The whole business of producing and selling LED lamps to retailer is of 3 years. The manufacturer offers a warranty period of 1 year to retailer and also credit period which is dependent on warranty period. When there is no warranty period, manufacturer gives credit period of 0.083 year. Produced items have been hold in a warehouse at cost of \$0.50 and sold at price of \$250. Due to offering credit period manufacturer has to loss interest at the rate of 2%. The main goal is to find the number of active and cold standby components and the number of cycles to maximize manufacturer's profit and system reliability with some constraints such that volume, cost and weight of the whole system should not exceed 60 *cubic cm*, \$50 and 350 gm respectively.

Values of input data are given in Tables 2 and 3.

In this example, the results are discussed comparing with active and mixed strategies. The results for active strategy are given in Table 4 and for mixed strategy in Table 5.

TABLE 4. Output value (Active strategy) for Example 1.

Variables (x_1, x_2, η)	Profit (Z)	System reliability (R)	Demand (D)
(1, 1, 2)	902125.50	0.7029	4210.73
(2, 1, 3)	851881.75	0.8098	5253.15
(3, 1, 3)	828043.00	0.8261	5443.56
(1, 6, 4)	792023.00	0.8479	5713.22
(2, 2, 4)	674486.50	0.9483	7210.37
(2, 6, 5)	565129.00	0.9768	7724.22

TABLE 5. Output value (Mixed strategy) for Example 1.

Variables $(x_1, x_2, y_1, y_2, \eta)$	Profit (Z)	System reliability (R)	Demand (D)
(1, 3, 1, 3, 5)	540284.00	0.9865	7909.38
(3, 1, 1, 2, 8)	513559.50	0.9962	8100.00
(1, 4, 2, 6, 7)	442503.50	0.9972	8119.67
(6, 3, 1, 1, 5)	434285.50	0.9997	8170.15
(10, 5, 2, 1, 4)	311651.00	1.0000	8176.20

TABLE 6. Output value (Active strategy) for Example 2.

Variables (x_1, x_2, η)	Profit (Z)	System reliability (R)	Demand (D)
(2, 3, 4)	1236814.00	0.9720	7493.69
(2, 4, 5)	1222380.00	0.9760	7568.31
(2, 5, 5)	1210195.00	0.9767	7581.46
(3, 3, 5)	1203673.00	0.9915	7866.06
(3, 4, 5)	1187511.50	0.9956	7947.17
(3, 5, 6)	1174570.50	0.9963	7961.11

Example 2 (When $t \leq W$, $s \geq s_1$):

This is an example related to Case-II of the crisp model when failure time t does not exceed the warranty period W and the selling price s lies after the certain value s_1. This example is same as Example 1, the only difference is in the form of demand function described in assumption (ix). Here, $t = 0.75$, $s = 280$, $s_1 = 270$ and $\beta = 0.1$. Also, the value of D_2 is obtained from the relation $D_2 = D_1(s_1 - s_0)^q$ which is proved by the following lemma.

Lemma 2. $D_2 = D_1(s_1 - s_0)^q$.

Proof. From the equation of demand function (assumption (ix)), at continuity point $s = s_1$, it is obtained that

$$D_1(s_1 - s_0)^q = D_2 e^{\beta(s_1 - s_1)}$$
$$\therefore D_2 = D_1(s_1 - s_0)^q$$

Hence the proof. □

For this example, the results of active and mixed strategies are given in Tables 6 and 7 respectively.

TABLE 7. Output value (Mixed strategy) for Example 2.

Variables $(x_1, x_2, y_1, y_2, \eta)$	Profit (Z)	System reliability (R)	Demand (D)
(2, 3, 1, 2, 8)	1172062.00	0.9984	8003.13
(2, 3, 1, 4, 6)	1147345.00	0.9988	8011.16
(6, 3, 1, 2, 8)	1096795.00	0.9999	8033.30
(6, 5, 2, 1, 4)	1063992.00	1.0000	8035.32

TABLE 8. Output value (Active strategy) for Example 3.

Variables (x_1, x_2, η)	Profit (Z)	System reliability (R)	Demand (D)
(1, 2, 2)	901133.25	0.7049	4227.32
(1, 4, 3)	874711.25	0.7556	4682.96
(2, 2, 3)	798589.75	0.8743	6064.30
(2, 4, 4)	688847.00	0.9373	7023.50
(3, 3, 5)	623750.50	0.9670	7542.55
(3, 4, 5)	579400.00	0.9810	7803.73
(4, 4, 6)	534262.00	0.9915	8006.94

TABLE 9. Output value (Mixed strategy) for Example 3.

Variables $(x_1, x_2, y_1, y_2, \eta)$	Profit (Z)	System reliability (R)	Demand (D)
(2, 3, 1, 1, 5)	550146.00	0.9925	8026.63
(6, 3, 1, 1, 6)	459438.00	0.9983	8142.00
(3, 3, 4, 1, 8)	456055.00	0.9995	8166.12
(6, 5, 2, 1, 4)	408847.50	0.9999	8174.18
(6, 6, 5, 2, 4)	325809.00	1.0000	8176.20

Example 3 (When $t > W$, $s_0 \leq s \leq s_1$):

This example is related to Case-III of the crisp model when failure time t is more than the warranty period W and the selling price s lies in the particular value (s_0 and s_1). This example is similar as Example 1, the only difference is $t > W$ here. So, the input data are same as Example 1, only difference is $t = 1.25$, $s = 250$, $s_0 = 220$, $q = 0.8$ and $\xi = 0.8$. Tables 8 and 9 depict the results for active and mixed strategies respectively.

Example 4 (When $t > W$, $s \geq s_1$):

This is the same example related to Case-IV of the crisp model when failure time t is more than the warranty period W and the selling price s exceeds the particular value s_1. The extra parameters from Example 1 are $t = 1.25$, $s = 280$, $s_1 = 270$, $\beta = 0.1$, $\xi = 0.8$ and $D_2 = D_1(s_1 - s_0)^q$. The respective results of active and mixed strategies are provided in Tables 10 and 11.

5.1.1. Comparisons from different aspects

– From Tables 4 and 5 of Example 1 it can be observed that active strategy gives more profit and less system reliability comparing with the mixed strategy. Actually, a mixed strategy is introduced to get more system reliability compared with active strategy which is fulfilled in this example.

TABLE 10. Output value (Active strategy) for Example 4.

Variables (x_1, x_2, η)	Profit (Z)	System reliability (R)	Demand (D)
(2, 5, 5)	1246004.00	0.9409	6943.10
(2, 6, 5)	1234979.50	0.9418	6958.33
(2, 7, 5)	1224180.50	0.9421	6963.41
(3, 4, 5)	1219008.50	0.9810	7662.86
(3, 5, 6)	1204253.50	0.9847	7733.73
(4, 4, 6)	1192447.50	0.9915	7866.07

TABLE 11. Output value (Mixed strategy) for Example 4.

Variables $(x_1, x_2, y_1, y_2, \eta)$	Profit (Z)	System reliability (R)	Demand (D)
(1, 4, 2, 6, 15)	1134532.50	0.9930	7895.62
(6, 5, 2, 1, 4)	1083940.50	0.9999	8033.31
(6, 6, 5, 2, 4)	1002561.00	1.0000	8035.32

– Similarly as Example 1, it is noticed from Tables 6 and 7 that the manufacturer's profit is decreased and the retailer's demand is increased with the increasing of system reliability in both of the active and mixed strategies in Example 2.
– Comparing Examples 1 and 2, it can be said that Example 2 gives better result in profit for both of the active and mixed strategies.
– From Tables 8 and 9 of Example 3, it is also noticed that with the increasing of system reliability, retailer's demand is increased but the manufacturer's profit is decreased which is significant because according to assumption (xii) production cost is increased with system reliability.
– Example 3 delivers almost similar results of Example 1 (reveals from Tables 8 and 9). To produce fully reliable items (reliability=1) in Example 3, profit is \$325809.00 which is less than the profit \$311651.00 in Example 1 whereas the demand is exactly same (8176.20 unit/year). The reason of getting more profit is that manufacturer earns repairing cost in Example 3 and total number of components of first type (10+2) which is more costly, is greater than Example 3 (6+5).
– Again for active strategy (Tables 8 and 6) with same reliability (0.9915), Example 3 gives less profit (\$534262.00) but more demand (8006.94 unit/year) comparing with Example 2 (\$1203673.00 and 7866.06 unit/year respectively). In Example 3, selling price is less than Example 2, so getting more demand is justified. Although manufacturer earns repairing cost here, but less selling price as well as more number of components (4+4) comparing with Example 2 (3+3) decrease the profit.
– In Example 4, the relationship between profit, system reliability and demand is same as the former examples. Tables 10 and 11 show almost same result with Example 2, because selling price is same here.
– For same reliability 1, Example 4 gives less demand (8035.32 unit/year) than Example 1 (8176.20 unit/year), because selling price is more here and gives more profit (\$1002561.00) than Example 1 (\$311651.00), (Tables 11 and 5). Although number of components is more here but selling price is also high and manufacturer earns repairing cost. So, giving more profit is justified.
– For the same reliability 0.9915 and almost same demand (7866.06 unit/year and 7866.07 unit/year), Example 4 gives less profit (\$1192447.50) than Example 2 (\$1203673.00) because number of components (4+4) is more here than Example 2 (3+3), (Tables 6 and 10). Similar explanation is applicable for same reliability 1 and same demand 8035.32 unit/year in mixed strategy (Tables 7 and 11).

– Example 3 and Example 4 give different profits (\$534262.00 and \$1192447.50) and different demand (8006.94 unit/year and 7866.07 unit/year) for same set of solutions and same reliability because of selling price and failure time (Tables 8 and 10). Similar explanation is appropriate for same set of solutions (6,6,5,2,4) and same reliability 1 in mixed strategy (Tables 9 and 11).

Henceforth, after overall comparisons it is observed that introducing mixed strategy provides more reliable products but manufacturer's profit is more in using active strategy. It is also noticed that Case-IV ($t > W,\ s \geq s_1$) gives maximum profit (\$1246004.00) out of these sets of solutions and mixed strategy produced fully reliable items according to Pareto optimal set of solutions. So, this is up to decision maker who can choose any solution from these sets.

5.1.2. Sensitivity analyses

In this subsection, some sensitivity analyses have been carried out to study the effects of changes in parameters and the figures regarding sensitivity analyses show the relationship for active strategy.

Sensitivity analysis 1:

Here, sensitivity analysis, based on Example 1, has been shown by changing the parameters, failure time (t) and warranty period (W). Table 12 shows set of two solutions for active strategy and one solution for mixed strategy.

– From Table 12 it is noticed that credit period (M) is decreased with increasing of W (when t is fixed) and remains same with increasing of t along with fixed W due to formulation structure of credit period (assumption (viii)).
– Again, it is observed from Table 12 and Figure 7(a,b) that manufacturer's profit (Z) is increased although demand (D) is decreased with increasing of W along with fixed t and also with increasing of t along with fixed W. When W is increased (M is decreased), D is decreased due to formulation structure (assumption (viii)) and also with decreasing of M, interest loss is also decreased and so profit is increased.
– Table 12 and Figure 7(c,d) show that system reliability (R) remains same with increasing of W when t is fixed and R is decreased with increasing of t for fixed W, since component reliabilities are also decreased with increasing of t.

Sensitivity analysis 2:

Here, sensitivity analysis for Example 1 has been performed by changing the values of the failure rate (λ_1, λ_2) and failure time (t).

– Table 13 and Figure 8 display that profit (Z) is increased and system reliability (R) is decreased with increasing of λ_1, λ_2 as well as t. Figure 8 depicts three increasing curves of Z with increasing of t for three set of values of λ_1 and λ_2, *i.e.*, (0.20,0.23), (0.22,0.25) and (0.24,0.27) and also three decreasing curves of R for the same changing values of λ_1, λ_2 and t. Table 13 also shows that for fixed t, with increasing of (λ_1, λ_2), component reliabilities (r_1, r_2) are decreased, hence system reliability is also decreased, again production cost is decreased for which profit is increased. Same reason is applicable for increasing values of t with fixed (λ_1, λ_2).

Sensitivity analysis 3:

By changing the values of parameter θ, a sensitivity analysis for Example 1 has been presented here. Table 14 shows set of four solutions.

– It is shown from Table 14 and Figure 9(a) that manufacturer's profit (Z) is decreased with increasing of θ because, for more damaged items, manufacturer has to repair more items in free of cost (manufacturer has to bear more repairing cost) and system reliability remains same with changing of θ because, rate of damaged item (θ) does not affect system reliability.

TABLE 12. Result of Example 1 for different values of t and W.

Parameters	Credit period	Active strategy						Mixed strategy							
W	M	x_1	x_2	η	Z	R	D	x_1	x_2	y_1	y_2	η	Z	R	D
$t = 0.55$															
1	0.073	2	1	3	811289.00	0.8602	5873.3	10	5	2	1	4	311633.50	1.0000	8176.2
	0.073	2	2	4	627060.50	0.9707	7610.5								
1.25	0.0705	2	1	3	817126.50	0.8602	5873.1	10	5	2	1	4	319795.50	1.0000	8176.0
	0.0705	2	2	4	634647.00	0.9707	7610.3								
1.5	0.068	2	1	3	822963.50	0.8602	5873.0	10	5	2	1	4	327956.50	1.0000	8175.8
	0.068	2	2	4	642234.50	0.9707	7610.1								
1.75	0.0655	2	1	3	828800.25	0.8602	5872.8	10	5	2	1	4	336117.00	1.0000	8175.6
	0.0655	2	2	4	649822.00	0.9707	7610.0								
$t = 0.65$															
1	0.073	2	1	3	833924.00	0.8349	5550.3	10	5	2	1	4	311640.00	1.0000	8176.2
	0.073	2	2	4	650328.50	0.9601	7417.8								
1.25	0.0705	2	1	3	839435.75	0.8349	5550.1	10	5	2	1	4	319801.00	1.0000	8176.0
	0.0705	2	2	4	657723.00	0.9601	7417.6								
1.5	0.068	2	1	3	844947.50	0.8349	5549.9	10	5	2	1	4	327963.50	1.0000	8175.8
	0.068	2	2	4	665116.50	0.9601	7417.4								
1.75	0.0655	2	1	3	850458.50	0.8349	5549.7	10	5	2	1	4	336124.00	1.0000	8175.6
	0.0655	2	2	4	672509.50	0.9601	7417.2								
$t = 0.75$															
1	0.073	2	1	3	851881.75	0.8098	5253.1	10	5	2	1	4	311651.00	1.0000	8176.2
	0.073	2	2	4	674486.50	0.9483	7210.4								
1.25	0.0705	2	1	3	857093.25	0.8098	5253.0	10	5	2	1	4	319812.50	1.0000	8176.0
	0.0705	2	2	4	681671.00	0.9483	7210.2								
1.5	0.068	2	1	3	862305.25	0.8098	5252.8	10	5	2	1	4	327973.50	1.0000	8175.8
	0.068	2	2	4	688855.50	0.9483	7210.0								
1.75	0.0655	2	1	3	867516.50	0.8098	5252.6	10	5	2	1	4	336135.00	1.0000	8175.6
	0.0655	2	2	4	696039.00	0.9483	7209.8								

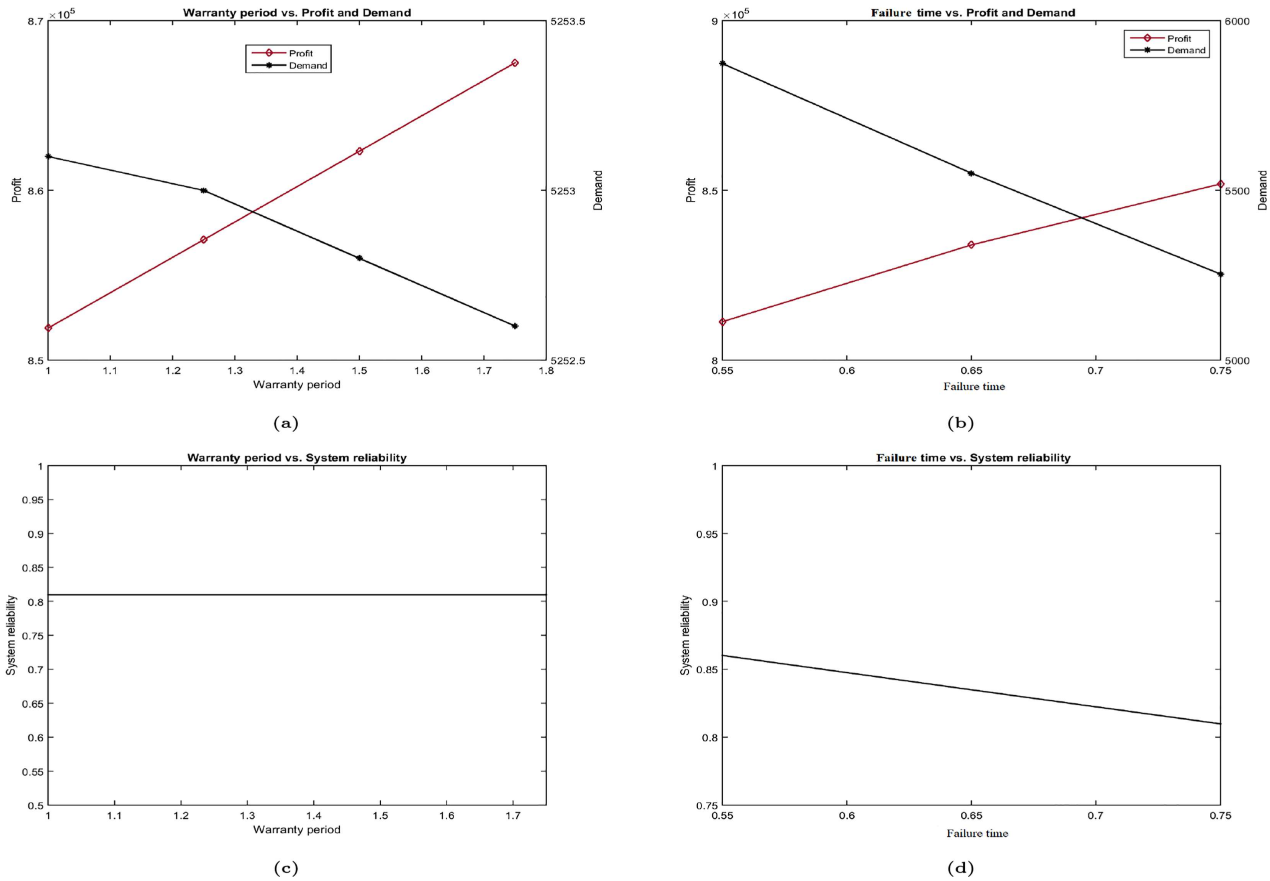


FIGURE 7. (a) Warranty period *vs.* Profit and Demand (b) Failure time *vs.* Profit and Demand (c) Warranty period *vs.* System reliability (d) Failure time *vs.* System reliability.

TABLE 13. Result of Example 1 for different values of λ_1, λ_2 and t.

Parameters	Component reliability		Active strategy							Mixed strategy								
(λ_1, λ_2)	r_1	r_2	x_1	x_2	η	Z	R	D	c_p	x_1	x_2	y_1	y_2	η	Z	R	D	c_p
$t = 0.55$																		
(0.20,0.23)	0.8958	0.8812	2	1	3	799409.75	0.8716	6027.1	174.32	10	5	2	1	4	311633.50	1.0000	8176.2	200.00
(0.22,0.25)	0.8860	0.8715	2	1	3	811289.00	0.8602	5873.3	172.04	10	5	2	1	4	311633.50	1.0000	8176.2	200.00
(0.24,0.27)	0.8763	0.8620	2	1	3	822095.75	0.8488	5724.7	169.76	10	5	2	1	4	311636.50	1.0000	8176.2	200.00
$t = 0.65$																		
(0.20,0.23)	0.8781	0.8611	2	1	3	822526.50	0.8483	5718.3	169.66	10	5	2	1	4	311636.50	1.0000	8176.2	200.00
(0.22,0.25)	0.8668	0.8500	2	1	3	833924.00	0.8349	5550.3	166.98	10	5	2	1	4	311640.50	1.0000	8176.2	200.00
(0.24,0.27)	0.8556	0.8390	2	1	3	844027.25	0.8215	5388.9	164.30	10	5	2	1	4	311645.50	1.0000	8176.2	200.00
$t = 0.75$																		
(0.20,0.23)	0.8607	0.8416	2	1	3	841358.75	0.8252	5432.8	165.04	10	5	2	1	4	311644.50	1.0000	8176.2	200.00
(0.22,0.25)	0.8479	0.8290	2	1	3	851881.75	0.8098	5253.1	161.96	10	5	2	1	4	311651.00	1.0000	8176.2	200.00
(0.24,0.27)	0.8353	0.8167	2	1	3	860934.50	0.7945	5082.7	158.90	10	5	2	1	4	311662.00	1.0000	8176.2	200.00

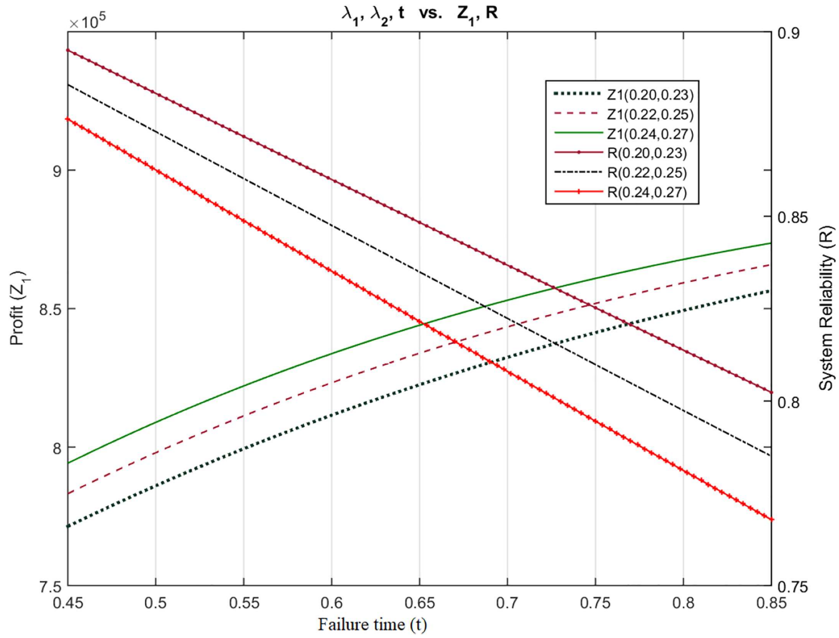
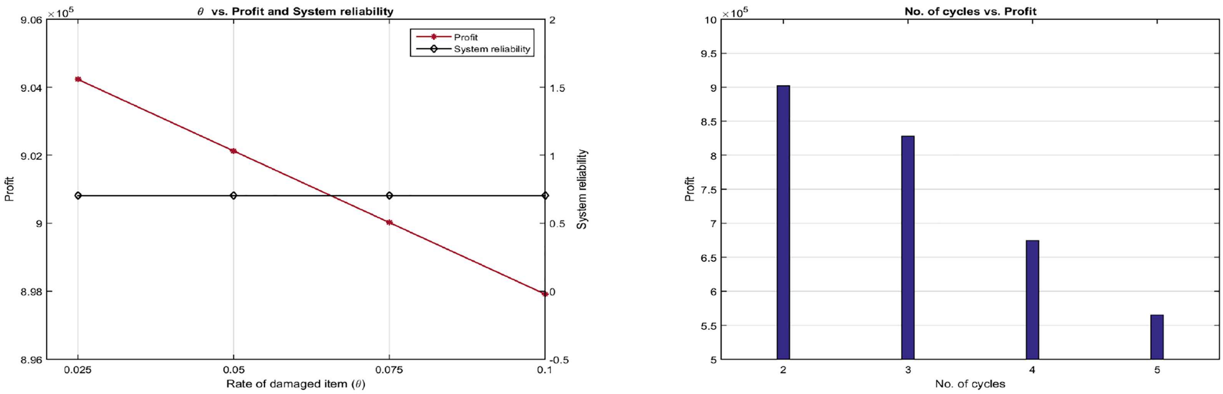


FIGURE 8. Failure rate and Failure time *vs.* Profit and System Reliability.

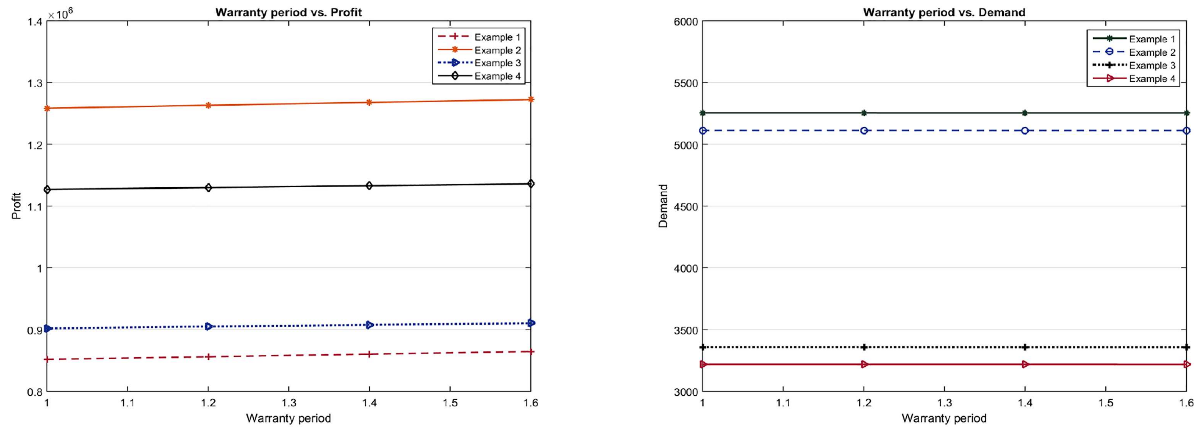
TABLE 14. Result of Example 1 for different values of θ.

Parameter	Active strategy					Mixed strategy						
θ	x_1	x_2	η	Z	R	x_1	x_2	y_1	y_2	η	Z	R
0.025	1	1	2	904231.00	0.7029	1	3	1	3	5	544238.50	0.9865
	3	1	3	830765.00	0.8261	3	1	1	2	8	517609.50	0.9962
	2	2	4	678091.50	0.9483	1	4	2	6	7	446563.50	0.9972
	2	6	5	568991.50	0.9768	6	3	1	1	5	438371.00	0.9997
0.05	1	1	2	902125.00	0.7029	1	3	1	3	5	540284.00	0.9865
	3	1	3	828043.00	0.8261	3	1	1	2	8	513559.50	0.9962
	2	2	4	674486.50	0.9483	1	4	2	6	7	442503.50	0.9972
	2	6	5	565129.00	0.9768	6	3	1	1	5	434285.50	0.9997
0.075	1	1	2	900020.00	0.7029	1	3	1	3	5	536329.00	0.9865
	3	1	3	825321.25	0.8261	3	1	1	2	8	509510.00	0.9962
	2	2	4	670881.00	0.9483	1	4	2	6	7	438443.50	0.9972
	2	6	5	561266.50	0.9768	6	3	1	1	5	430200.50	0.9997
0.1	1	1	2	897914.50	0.7029	1	3	1	3	5	532374.50	0.9865
	3	1	3	822599.25	0.8261	3	1	1	2	8	505460.50	0.9962
	2	2	4	667276.00	0.9483	1	4	2	6	7	434383.50	0.9972
	2	6	5	557404.00	0.9768	6	3	1	1	5	426115.00	0.9997



(a) (b)

FIGURE 9. (a) Rate of damaged item *vs.* Profit and System Reliability (b) Number of cycles *vs.* Profit.



(a) **(b)**

FIGURE 10. (a) Warranty period *vs.* Profit (b) Warranty period *vs.* Demand (for all examples).

– Table 14 and Figure 9(b) depict that for fixed θ (0.05) manufacturer's profit is decreased with increasing of number of components and number of cycles because, for more components, components' purchasing cost and holding cost is increased along with more ordering cost for more cycles. Again, it is also noticed from Table 14 that with increasing of number of components, R is also increased and for same number of components 4 (3+1 and 2+2), the combination of 2+2 gives more system reliability because, second type component is less reliable than first type ($\lambda_1 = 0.22, \lambda_2 = 0.25, t = 0.75$) and here number of second type component (*i.e.*, 2) is more comparing with the combination of 3+1.

Sensitivity analysis 4:

Here, some sensitivity analyses have been drawn in Tables 15–18 and Figures 10–13 by comparing all examples (Examples 1-4) with the changing values of W, λ_1, λ_2 and θ.

TABLE 15. Result of all examples for different values of W.

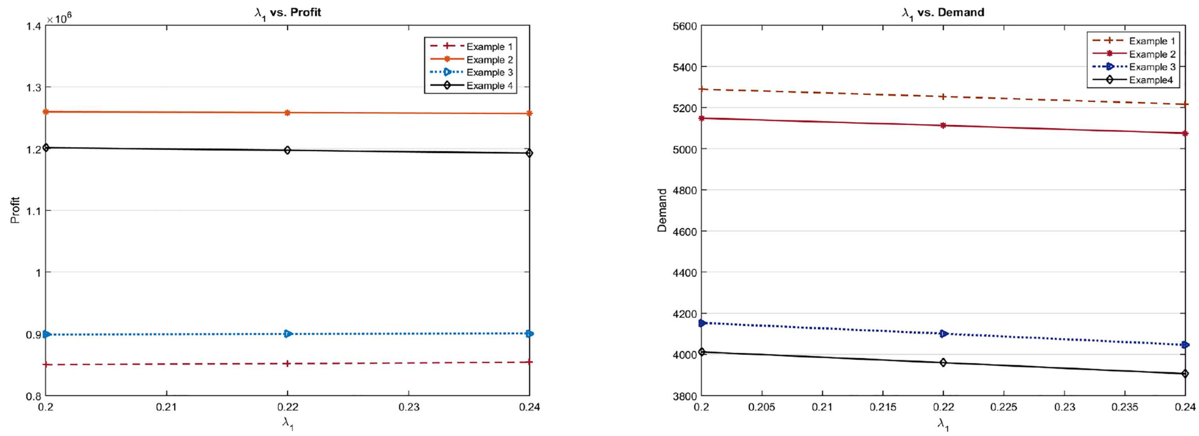
W	x_1	x_2	η	Z	R	D	Example
1	2	1	3	851880	0.8098	5253.7	1
	2	1	3	1258500	0.8098	5112.8	2
	2	1	3	902110	0.5797	3358.9	3
	2	1	3	1126900	0.5797	3218.0	4
1.2	2	1	3	856050	0.8098	5253.6	1
	2	1	3	1263200	0.8098	5112.7	2
	2	1	3	904880	0.5797	3358.8	3
	2	1	3	1129900	0.5797	3217.9	4
1.4	2	1	3	860220	0.8098	5253.4	1
	2	1	3	1267900	0.8098	5112.5	2
	2	1	3	907650	0.5797	3358.6	3
	2	1	3	1133000	0.5797	3217.7	4
1.6	2	1	3	864390	0.8098	5253.3	1
	2	1	3	1272500	0.8098	5112.4	2
	2	1	3	910410	0.5797	3358.5	3
	2	1	3	1136000	0.5797	3217.6	4

TABLE 16. Result of all examples for different values of λ_1.

λ_1	x_1	x_2	η	Z	R	D	c_p	Example
0.20	2	1	3	849880	0.8129	5289.2	162.59	1
	2	1	3	1259800	0.8129	5148.3	162.59	2
	2	1	3	898970	0.6958	4152.8	139.16	3
	2	1	3	1201500	0.6958	4011.9	139.16	4
0.22	2	1	3	851880	0.8098	5253.7	161.97	1
	2	1	3	1258500	0.8098	5112.8	161.97	2
	2	1	3	899790	0.6893	4100.7	137.86	3
	2	1	3	1197300	0.6893	3959.8	137.86	4
0.24	2	1	3	853960	0.8065	5216.1	161.31	1
	2	1	3	1257100	0.8065	5075.2	161.31	2
	2	1	3	900530	0.6825	4046.8	136.49	3
	2	1	3	1192800	0.6825	3905.9	136.49	4

– From Table 15 and Figure 10(a), it is observed that when warranty period (W) is fixed, manufacturer's profit (Z) is more for Example 2, then for Examples 4,3 and 1 respectively and the system reliability (R) is same for the couple of examples, Examples 1,2 and Examples 3,4. It is also seen that with the increasing of warranty period, profit is increased and system reliability remains same for each particular example.
– Again, Table 15 and Figure 10(b) reveals that demand (D) is decreased with the increasing of warranty period (W) for each example.

– Table 16 and Figure 11(a,b) depict that profit (Z) is increased for Examples 1 and 3 and decreased for Examples 2 and 4 with the increasing of failure rate (λ_1) and also demand (D) is decreased for each example. When λ_1 is fixed, manufacturer will be more profitable for Examples 2,4,3,1 serially.



(a) **(b)**

FIGURE 11. (a) λ_1 *vs.* Profit (b) λ_1 *vs.* Demand (for all examples).

TABLE 17. Result of all examples for different values of λ_2.

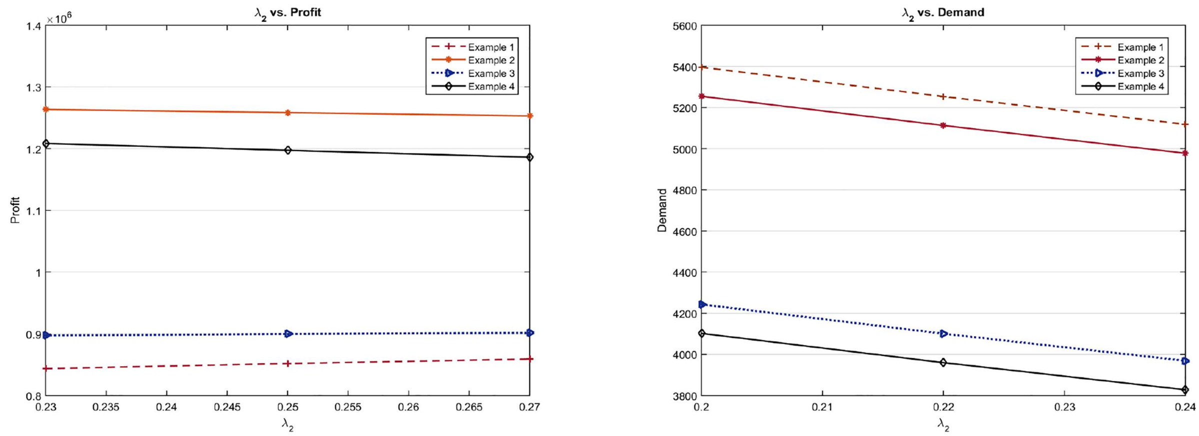
λ_2	x_1	x_2	η	Z	R	D	c_p	Example
0.23	2	1	3	843630	0.8221	5395.8	164.42	1
	2	1	3	1263500	0.8221	5253.9	164.42	2
	2	1	3	897340	0.7068	4243.0	141.35	3
	2	1	3	1208500	0.7068	4102.1	141.35	4
0.25	2	1	3	851880	0.8098	5253.7	161.97	1
	2	1	3	1258500	0.8098	5112.8	161.97	2
	2	1	3	899790	0.6893	4100.7	137.86	3
	2	1	3	1197300	0.6893	3959.8	137.86	4
0.27	2	1	3	859120	0.7978	5118.7	159.56	1
	2	1	3	1253200	0.7978	4977.8	159.56	2
	2	1	3	901450	0.6723	3968.9	134.46	3
	2	1	3	1186200	0.6723	3828.0	134.46	4

– Table 17 and Figure 12(a,b) indicate that with the increasing of failure rate (λ_2), profit (Z) is increased for Examples 1,3 and decreased for Examples 2,4 and also demand (D) is decreased for each example. When λ_2 is fixed, Example 2 gives better profit, then Examples 4,3,1 respectively give.

– From Table 18 and Figure 13, it is observed that with the increasing of rate of damaged item (θ), profit (Z) is decreased for Examples 1,2 and increased for Examples 3,4. When θ is fixed, Example 2 is the best profit case, then Examples 4,1,3 respectively.

5.2. Results for fuzzy model

In this subsection, six numerical examples have been illustrated for the fuzzy model. Here, failure time is determined from the failure rate by using fuzzy logic. The triangular fuzzy numbers of "Low"(L), "Medium"(Med) and "High"(H) for the failure rate λ_i and the failure time t_i for the Mamdani method has been considered as follows:



(a) **(b)**

FIGURE 12. (a) λ_2 *vs.* Profit (b) λ_2 *vs.* Demand (for all examples).

TABLE 18. Result of all examples for different values of θ.

θ	x_1	x_2	η	Z	R	D	c_p	Example
0.025	1	1	2	904230	0.7029	4211.0	140.59	1
	1	1	2	1261100	0.8098	5112.8	161.97	2
	1	1	2	899050	0.6893	4100.7	137.86	3
	1	1	2	1185700	0.6723	3828.0	134.46	4
0.05	1	1	2	902130	0.7029	4211.0	140.59	1
	1	1	2	1258500	0.8098	5112.8	161.97	2
	1	1	2	899790	0.6893	4100.7	137.86	3
	1	1	2	1186200	0.6723	3828.0	134.46	4
0.075	1	1	2	900020	0.7029	4211.0	140.59	1
	1	1	2	1256000	0.8098	5112.8	161.97	2
	1	1	2	900520	0.6893	4100.7	137.86	3
	1	1	2	1186800	0.6723	3828.0	134.46	4
0.1	1	1	2	897910	0.7029	4211.0	140.59	1
	1	1	2	1253400	0.8098	5112.8	161.97	2
	1	1	2	901250	0.6893	4100.7	137.86	3
	1	1	2	1187300	0.6723	3828.0	134.46	4

Failure rate: $Low = (0.1, 0.2, 0.4)$, $Med = (0.2, 0.4, 0.6)$ and $High = (0.4, 0.6, 0.9)$
Failure time: $Low = (0.5, 1.5, 2)$, $Med = (1.5, 2, 2.5)$ and $High = (2, 2.5, 2.8)$

The extra parametric values for the Sugeno method are $a_1 = 0.3, a_2 = 0.5, a_3 = 0.7$ and $b_1 = b_2 = b_3 = 1.2$ which are related to equation (6). For the Rule 1 of the Tsukamoto method, *i.e.*, for the monotonically decreasing function of logistic curve, the parameters are $k_1 = -5$ and $t_0 = 2$; for the Rule 2, *i.e.*, for the monotonically increasing function, $k_1 = 5$ and $t_0 = 2$ and Rule 3 is not applicable for $\lambda_1 = 0.22$ and $\lambda_2 = 0.25$. Remaining parametric values are same as the previous examples of crisp model described in Tables 2 and 3.

The optimum results for fuzzy model have been obtained by applying the algorithm stated in Section 4. For the input values $\lambda_1 = 0.22$ and $\lambda_2 = 0.25$, the corresponding output values of failure time t_1 and t_2 have been determined by three fuzzy inference techniques as $\{t_1, t_2\} = \{2.3589, 2.2756\}$ (for Mamdani method),

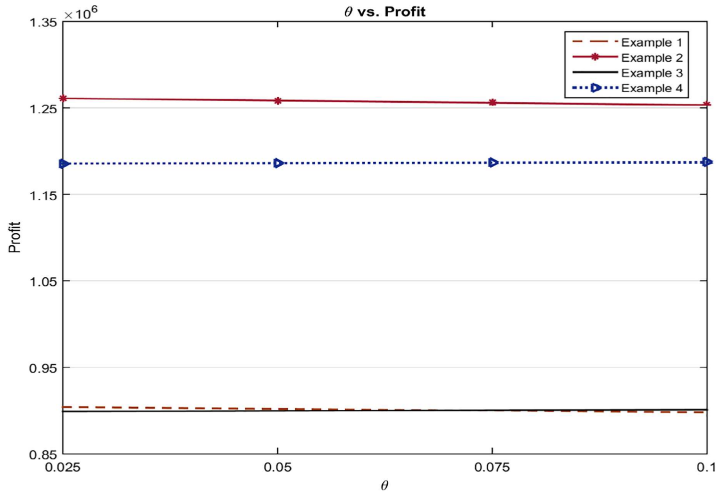


FIGURE 13. θ *vs.* Profit for all examples.

$\{t_1, t_2\} = \{1.1296, 1.1125\}$ (for Sugeno method) and $\{t_1, t_2\} = \{1.5605, 1.7802\}$ (for Tsukamoto method). So, the corresponding value of t, where $t = min\{t_1, t_2\}$, for these three fuzzy inference techniques are $t = 2.2756$, $t = 1.1125$ and $t = 1.5605$ respectively. It is observed that all the values of corresponding failure time t exceeds the warranty period ($W = 1$), *i.e.*, $t > W$. So, in this subsection the discussion should be made only for the two cases stated in the crisp model, *i.e.*, Case-III (When $t > W$, $s_0 \leq s \leq s_1$) and Case-IV (When $t > W$, $s \geq s_1$). Consequently, here six numerical examples have been discussed for these two cases each of the three fuzzy inference methods.

Example 5 (Mamdani method when $s_0 \leq s \leq s_1$):

This example is related to the Example 3 of the crisp model. Here, failure time t is obtained by the Mamdani fuzzy inference. The different parameters from the Example 3 is $t_1 = 2.3589$, $t_2 = 2.2756$ and $t = min\{t_1, t_2\} = 2.2756$. The corresponding result for active strategy is shown in Table 19 and for mixed strategy is shown in Table 20.

Example 6 (Mamdani method when $s \geq s_1$):

This example is related to the Example 4 of the crisp model. Here also the failure time t is obtained by the Mamdani fuzzy inference. The parametric values are same as Example 5 but here selling price s exceeds the certain value s_0, *i.e.*, $D_2 = D_1(s_1 - s_0)^q$ (according to Lem. 2). Similar as Example 5, here, $t_1 = 2.3589$, $t_2 = 2.2756$ and $t = 2.2756$. The corresponding result for active strategy is shown in Table 21 and for mixed strategy, it is shown in Table 22.

Example 7 (Sugeno method when $s_0 \leq s \leq s_1$):

This example is similar as Example 5, but here, failure time t is obtained by the Sugeno method. The different parameters from the Example 5 is $t_1 = 1.1296$, $t_2 = 1.1125$ and $t = 1.1125$. The corresponding result for active strategy is shown in Table 19 and for mixed strategy, it is shown in Table 20.

Example 8 (Sugeno method when $s \geq s_1$):

This example is same as Example 7, but here, selling price s exceeds the certain value s_0. Here also, the failure time t is obtained by the Sugeno method. The parametric values are same as Example 7, but here $D_2 = D_1(s_1 - s_0)^q$. Similar as Example 7, here also, $t_1 = 1.1296$, $t_2 = 1.1125$ and $t = 1.1125$. The corresponding result for active strategy is shown in Table 21 and for mixed strategy, it is shown in Table 22.

TABLE 19. Comparative results of active strategy when $s_0 \leq s \leq s_1$.

Mamdani method					Sugeno method					Tsukamoto method				
x_1	x_2	η	Z	R	x_1	x_2	η	Z	R	x_1	x_2	η	Z	R
1	3	2	898807.38	0.5566	1	1	1	910963.38	0.5928	1	2	2	906231.88	0.6354
2	2	2	894042.00	0.6859	2	1	2	894764.75	0.7215	1	8	4	860754.50	0.7093
3	2	3	864943.50	0.7622	2	2	3	771978.00	0.8967	2	2	3	845997.50	0.8200
4	3	5	733142.25	0.8962	2	5	5	649720.50	0.9521	2	7	5	694833.00	0.9152
5	4	6	603339.00	0.9554	3	5	6	547163.00	0.9889	3	5	6	588995.00	0.9720
6	5	7	513579.50	0.9810	5	7	7	456077.50	0.9995	4	4	6	558694.00	0.9821
7	6	5	452697.50	0.9919	6	8	7	423544.50	0.9999	12	6	5	337573.00	0.9989

TABLE 20. Comparative results of mixed strategy when $s_0 \leq s \leq s_1$.

Mamdani method							Sugeno method							Tsukamoto method						
x_1	x_2	y_1	y_2	η	Z	R	x_1	x_2	y_1	y_2	η	Z	R	x_1	x_2	y_1	y_2	η	Z	R
1	1	1	1	7	855786.50	0.8023	6	2	4	1	4	426591.00	0.9940	1	1	1	2	11	702696.00	0.9343
3	1	1	1	7	766140.75	0.8775	6	4	2	1	4	421985.50	0.9998	2	1	1	2	11	609169.50	0.9728
2	2	1	2	4	628940.00	0.9593	6	5	2	1	4	408716.00	0.9999	1	2	2	3	8	537643.50	0.9924
1	2	2	4	1	492100.00	0.9997	10	5	2	1	4	332302.50	1.0000	3	2	5	2	11	436794.00	0.9995

Example 9 (Tsukamoto method when $s_0 \leq s \leq s_1$):

This example is related to the Example 5, but here, failure time t is obtained by the Tsukamoto method. The different parameters from the Example 5 is $t_1 = 1.5605$, $t_2 = 1.7802$ and $t = 1.5605$. The corresponding result for active strategy is shown in Table 19 and for mixed strategy, it is shown in Table 20.

Example 10 (Tsukamoto method when $s \geq s_1$):

This example is same as Example 9, but here, selling price s exceeds the certain value s_0. Here also, the failure time t is obtained by the Tsukamoto method. The parametric values are same as Example 9 but here $D_2 = D_1(s_1 - s_0)^q$. Like Example 9, $t_1 = 1.5605$, $t_2 = 1.7802$ and $t = 1.5605$ here. The corresponding result for active strategy is shown in Table 21 and for mixed strategy, it is shown in Table 22.

5.2.1. Discussion and comparison

- From Table 19, it is observed that for active strategy, when selling price s lies in the particular value (between s_0 and s_1), Sugeno method gives better system reliability (0.9999) as well as better profit ($910963.38) compared to the rest of the two methods, *i.e.*, Mamdani method and Tsukamoto method.
- For the mixed strategy, when selling price s lies between s_0 and s_1, it is observed from Table 20 that the Sugeno method gives better system reliability 1 where the corresponding profit is very less ($332302.50) but the Mamdani method gives better profit ($855786.50) with corresponding reliability 0.8023.
- When the selling price s exceeds the particular value s_1, for active strategy comaring the three methods it is observed from Table 21 that the Sugeno method gives best reliability 1 and the Tsukamoto method gives better profit ($1226134.50)
- For mixed strategy, when the selling price s exceeds the particular value s_1, Table 22 reveals that the Tsukamoto method not only gives the best reliability 1 with corresponding profit $1042928.50 but also the better profit $1242685.50 with corresponding reliability 0.9493 compared to the other two methods, *i.e.*, Mamdani and Sugeno methods.

Table 21. Comparative results of active strategy when $s \geq s_1$.

Mamdani method					Sugeno method					Tsukamoto method				
x_1	x_2	η	Z	R	x_1	x_2	η	Z	R	x_1	x_2	η	Z	R
4	5	9	1203123.00	0.9609	3	5	6	1200743.00	0.9889	4	3	9	1226134.50	0.9594
5	5	6	1176609.00	0.9753	4	4	6	1189846.00	0.9943	4	4	6	1200459.00	0.9821
5	7	7	1142078.50	0.9877	4	6	6	1162199.00	0.9976	4	5	6	1182321.50	0.9894
6	7	7	1118024.50	0.9934	5	5	7	1154897.00	0.9987	5	5	7	1159325.00	0.9944
6	9	8	1091122.00	0.9957	6	6	7	1123062.00	0.9997	6	7	8	1111560.50	0.9990
8	9	9	1050558.50	0.9989	6	8	8	1098289.00	0.9999	7	7	8	1092661.50	0.9995
9	11	9	1006666.00	0.9997	7	8	8	1079768.50	1.0000	7	9	8	1067736.00	0.9998

Table 22. Comparative results of mixed strategy when $s \geq s_1$.

Mamdani method							Sugeno method							Tsukamoto method						
x_1	x_2	y_1	y_2	η	Z	R	x_1	x_2	y_1	y_2	η	Z	R	x_1	x_2	y_1	y_2	η	Z	R
3	1	1	2	9	1229888.50	0.9518	2	3	1	1	5	1207884.00	0.9946	1	3	1	2	6	1242685.50	0.9493
6	2	1	2	8	1144042.00	0.9855	2	3	1	3	4	1181492.00	0.9962	2	3	1	2	6	1201211.00	0.9884
6	3	1	2	8	1122863.00	0.9944	2	3	1	4	14	1166882.50	0.9970	2	3	2	1	7	1190748.50	0.9932
10	5	2	1	5	1012976.50	0.9982	6	3	1	2	8	1116943.00	0.9996	2	3	2	2	6	1176345.00	0.9957
14	6	5	2	4	853039.00	0.9997	6	5	2	1	4	1083886.00	0.9999	6	5	2	1	4	1084194.50	0.9997
9	15	3	8	6	801606.00	1.0000	10	5	2	1	5	1010493.50	1.0000	8	6	1	2	8	1042928.50	1.0000

6. Managerial insights

In this section, some managerial insights have been drawn.

– From this article, decision-makers will be helpful to select solutions that best meet their priorities, whether they focus on profit or system reliability.
– This study helps manufacturers understand how different parameters affect profit and system reliability, allowing them to choose the proper value of parameters.
– From the numerical analysis, it is observed that fuzzy model gives better profit compared to the crisp model most of the situations when failure time exceeds the warranty period not only for the active strategy but also for the mixed strategy excepts the situation for active strategy when selling price exceeds the particular value. So, from the overall discussion, the manufacturer may go through the fuzzy model to get more profit.
– From this study, it is observed that to get more profit the manufacturer has to set high warranty period and also has to produce the product with high failure time for Case-I of the crisp model.
– This study informs about the most appropriate fuzzy inference method among Mamdani, Sugeno and Tsukamoto.

7. Conclusion and future research scope

This study deals with a production-reliability-inventory model (PRIM) of a single manufacturer and a single retailer in which different types of raw-materials have been used as component in a series-parallel system to produce items (devices) which are sold throughout the business period of 3 years. Here, raw-materials are procured in some cycles of equal length and the manufacturer offers warranty period and credit period to the retailer. This paper consists of two scenarios: an RAP with crisp structure and the other with fuzzy logic. Here, the crisp model contains four cases: Case-I, when failure time (t) of the item is less or equal to warranty period (W) and the selling price (s) lies between s_0 and s_1; Case-II, when $t \leq W$ and $s \geq s_1$; Case-III, when $t > W$ and $s_0 \leq s \leq s_1$ and Case-IV, when $t > W$ and $s \geq s_1$. Again, the fuzzy model contains the situation when failure time is depending on the failure rate following IF-THEN rule-based fuzzy logic and solved by three fuzzy inference techniques: Mamdani method, Sugeno method and Tsukamoto method.

In the present day, customers are very much conscious about reliability of the product and also, selling price is a crucial factor for demand. So, reliability, selling price and credit period dependent demand has been considered here. The main goal of this study is to determine the number of components used in the system and the number of cycles of collecting raw-materials which maximize manufacturer's profit as well as reliability of the product. Since the proposed problem is a complicated multi-objective optimization problem, so NSGA-II has been applied for solving such type of problem.

Numerically, four examples have been evaluated for four different cases of the crisp model and six examples for the fuzzy model which have been solved by using NSGA-II. The comparisons from different aspects of mixed strategy, active strategy; different cases; different fuzzy inference techniques and also between crisp and fuzzy models have been illustrated here. Some sensitivity analyses and managerial insights have been performed on the basis of changing different parameters. After dealing with this model, we can conclude that mixed strategy gives more system reliability compared to active strategy and profit gives the better result when selling price is more ($\geq s_1$). Observing the Pareto optimal solutions for all examples of the crisp model, it can be concluded that Example 4 (example related to Case-IV of the crisp model) gives maximum profit with active strategy and mixed strategy gives the solutions of producing more reliable products. Also, fuzzy model gives better profit compared to the crisp model in most of the situations when failure time exceeds the warranty period not only for the active strategy but also for the mixed strategy. In case of business, maximum profit solution is suitable but system reliability is less here. So, this is up to decision maker who can choose any solution from the Pareto optimal set of solutions.

For future research work, one can modify this model by using stochastic demand and the demand depending on several factors, time value of money, inflation, different levels of credit period etc. Another future scope is that new metaheuristic algorithms can be developed to solve this model.

Appendix A.

Raw-materials' holding cost: Raw-materials' holding cost is calculated with the help of the inventory of the raw-materials. In the proposed model, suppose a manufacturer collects Q_{r_i} quantity of different types (n types) of raw-materials in some cycles (η) of equal length T_0 and total time of collecting raw-materials *i.e.*, the total production time is T_1, $\left(T_0 = \dfrac{T_1}{\eta}\right)$. Here, the rate of raw-materials demand is denoted by D_{r_i}. We have considered that the production cell contains n subsystems and each subsystem contains x_i number of active components and y_i number of cold standby components. n types of components are used in n subsystems and each subsystem contains only one type of components (*i.e.*, i^{th} subsystem contains i^{th} type of components). So, $\sum_{i=1}^{n}(x_i + y_i)$ number of components are needed to produce a single item.

Inventory level $I_{r_i}(t)$ of the i^{th} type of raw-materials at time t Figure 1(a) is described by

$$\frac{dI_{r_i}(t)}{dt} = -D_{r_i}, \quad 0 \leq t \leq T_0 \tag{A.1}$$

with the following boundary conditions

$$I_{r_i}(0) = Q_{r_i} \tag{A.2}$$
$$I_{r_i}(T_0) = 0 \tag{A.3}$$

Using boundary conditions (A.2) and (A.3), solving equation (A.1) it is obtained that

$$\begin{aligned} I_{r_i}(t) &= -D_{r_i}t + Q_{r_i} \\ Q_{r_i} &= D_{r_i}T_0 \end{aligned}$$

Now, the holding cost of i^{th} type of raw-materials for η cycle is given by

$$\eta c_{4_i} \int_0^{T_0} I_{r_i}(t)\,dt = \eta c_{4_i}\Big[-\frac{1}{2}D_{r_i}T_0^2 + Q_{r_i}T_0\Big]$$

So, total raw-materials' holding cost, RHC is given by

$$RHC = \sum_{i=1}^{n} \eta c_{4_i}\Big[-\frac{1}{2}D_{r_i}T_0^2 + Q_{r_i}T_0\Big] = \sum_{i=1}^{n} \frac{1}{2}\eta c_{4_i} P(x_i+y_i)T_0^2$$

Appendix B.

Item's holding cost: To calculate the items' holding cost, inventory of finished product should be studied. At the production rate P, the manufacturer produces items during $t=0$ to $t=T_1$ and sells them all over the business time, *i.e.*, from $t=0$ to $t=T$ Figure 1(c), according to the demand rate D which follows the condition given in assumption (ix).

So, manufacturer's inventory level $I_f(t)$ of finished product is given by

$$\frac{dI_f(t)}{dt} = \begin{cases} P-D & ;0\le t\le T_1 \\ -D & ;T_1\le t\le T \end{cases} \tag{B.4}$$

with boundary conditions

$$I_f(0) = I_f(T) = 0 \tag{B.5}$$

Using boundary conditions (B.5), solving equation (B.4) it is obtained that

$$I_f(t) = \begin{cases} (P-D)t & ;0\le t\le T_1 \\ D(T-t) & ;T_1\le t\le T \end{cases}$$

By considering the continuity at time $t=T_1$, it is obtained that

$$T_1 = \frac{DT}{P}$$

So, finished items' holding cost, IHC is given by

$$IHC = c_5\Big[\int_0^{T_1} I_f(t)\,dt + \int_{T_1}^{T} I_f(t)\,dt\Big]$$
$$\therefore IHC = \frac{1}{2}c_5\frac{DT^2}{P}\Big(P-D\Big)$$

Appendix C.

$$R(t;x,y) = \prod_{i=1}^{n}\Big[\Big\{1-(1-e^{-\lambda_i t})^{x_i}\Big\} + \rho\int_0^t x_i(1-e^{-\lambda_i t_1})^{x_i-1}\lambda_i e^{-\lambda_i t_1}e^{-\lambda_i(t-t_1)}\,dt_1$$
$$+\sum_{l=1}^{y_i-1}\rho^{l+1}\int_0^t\int_{t_1}^t x_i(1-e^{-\lambda_i t_1})^{x_i-1}\lambda_i e^{-\lambda_i t_1}\lambda_i e^{-\lambda_i u}e^{-\lambda_i(t-u)}\,du\,dt_1\Big]$$

$$
\begin{aligned}
= \prod_{i=1}^{n} \Bigg[\Big\{ 1 - (1 - e^{-\lambda_i t})^{x_i} \Big\} + \rho x_i \lambda_i e^{-\lambda_i t} \int_0^t (1 - e^{-\lambda_i t_1})^{x_i - 1} \, dt_1 \\
+ \sum_{l=1}^{y_i - 1} \rho^{l+1} x_i \lambda_i^2 e^{-\lambda_i t} \int_0^t \int_{t_1}^t (1 - e^{-\lambda_i t_1})^{x_i - 1} e^{-\lambda_i t_1} \, du \, dt_1 \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \prod_{i=1}^{n} \Bigg[\Big\{ 1 - (1 - e^{-\lambda_i t})^{x_i} \Big\} + \rho x_i \lambda_i e^{-\lambda_i t} \Big\{ t - \frac{1}{\lambda_i} \sum_{p=1}^{x_i - 1} \frac{(1 - e^{-\lambda_i t})^p}{p} \Big\} \\
+ \sum_{l=1}^{y_i - 1} \rho^{l+1} x_i \lambda_i^2 e^{-\lambda_i t} \Big\{ \frac{t}{\lambda_i x_i} - \frac{1}{\lambda_i^2 x_i} \sum_{p=1}^{x_i} \frac{(1 - e^{-\lambda_i t})^p}{p} \Big\} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \prod_{i=1}^{n} \Bigg[\Big\{ 1 - (1 - e^{-\lambda_i t})^{x_i} \Big\} + \rho x_i \lambda_i t e^{-\lambda_i t} - \rho x_i e^{-\lambda_i t} \sum_{p=1}^{x_i - 1} \frac{(1 - e^{-\lambda_i t})^p}{p} \\
+ \Big(\sum_{l=1}^{y_i - 1} \rho^{l+1} \Big) e^{-\lambda_i t} \Big\{ \lambda_i t - \sum_{p=1}^{x_i} \frac{(1 - e^{-\lambda_i t})^p}{p} \Big\} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \prod_{i=1}^{n} \Bigg[\Big\{ 1 - (1 - e^{-\lambda_i t})^{x_i} \Big\} + \rho x_i \lambda_i t e^{-\lambda_i t} - \rho x_i e^{-\lambda_i t} \sum_{p=1}^{x_i - 1} \frac{(1 - e^{-\lambda_i t})^p}{p} \\
+ \Big(\frac{\rho^2 - \rho^{y_i + 1}}{1 - \rho} \Big) e^{-\lambda_i t} \Big\{ \lambda_i t - \sum_{p=1}^{x_i} \frac{(1 - e^{-\lambda_i t})^p}{p} \Big\} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \prod_{i=1}^{n} \Bigg[1 - (1 - e^{-\lambda_i t})^{x_i} + \rho x_i \lambda_i t e^{-\lambda_i t} - \Big(\rho x_i + \frac{\rho^2 - \rho^{y_i + 1}}{1 - \rho} \Big) e^{-\lambda_i t} \sum_{p=1}^{x_i - 1} \frac{(1 - e^{-\lambda_i t})^p}{p} \\
+ \Big(\frac{\rho^2 - \rho^{y_i + 1}}{1 - \rho} \Big) e^{-\lambda_i t} \Big\{ \lambda_i t - \frac{(1 - e^{-\lambda_i t})^{x_i}}{x_i} \Big\} \Bigg]
\end{aligned}
$$

Appendix D.

Fuzzy inference with fuzzy logic:

The term "inference" is associated with the process of gaining new information by applying previously acquired knowledge and the process of translating a set of input values into a set of output values by applying fuzzy logic ideas is known as fuzzy inference [12]. It is also known as the IF-THEN type rule. In this process, the given knowledge of a fact, called premise/hypothesis/antecedent, allows us to derive another truth known as a conclusion/consequent. For example, "IF x_1 is a_1 THEN x_2 is a_2", where x_1 is antecedent and x_2 is consequent. There are three well-known fuzzy inference techniques in the literature such as Mamdani method, Sugeno method and Tsukamoto method.

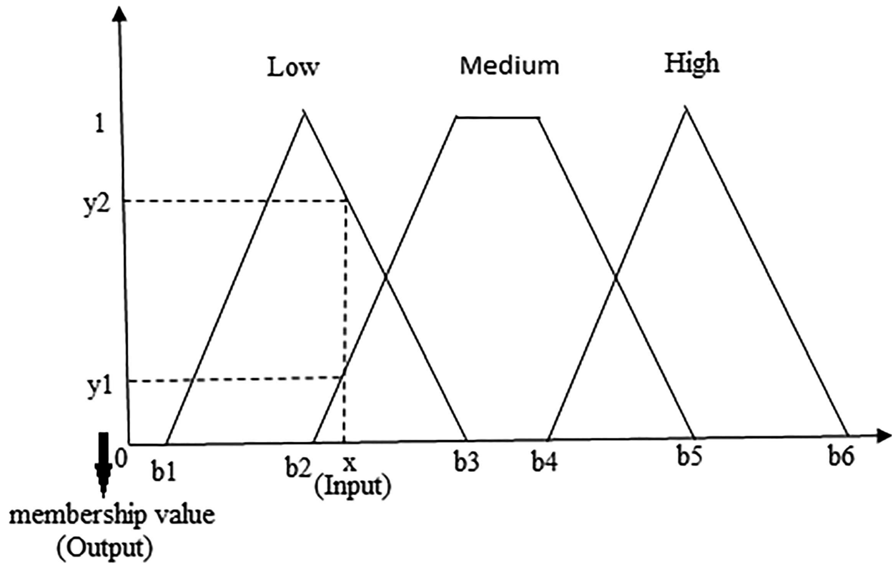


FIGURE D.1. Fuzzification of input values.

(1) Mamdani method:

Among the existing fuzzy inference techniques Mamdani method has been the most widely utilized methodology. Ebrahim Mamdani proposed this approach in 1975 [31]. Four main steps of Mamdani fuzzy inference process are

(i) Fuzzification of input values: Fuzzification of inputs is done based on membership functions. The input value of an antecedent must correspond to one or more linguistic fuzzy sets (for example, "Low", "Medium", "High" etc.) with membership values lies between 0 and 1 Figure D.1. Prior decisions are made regarding the domain of the function and the shape of the membership function (triangular, trapezoidal, gaussian etc.).

(ii) Rule evaluation and rule strength calculation: The IF-THEN type rule of the Mamdani method can be represented as follows:

Rule 1: IF x_1 is $\widetilde{A_1}$ THEN y is $\widetilde{B_1}$
Rule 2: IF x_1 is $\widetilde{A_1}$ AND x_2 is $\widetilde{A_2}$ THEN y is $\widetilde{B_2}$
Rule 3: IF x_1 is $\widetilde{A_1}$ OR x_2 is $\widetilde{A_2}$ THEN y is $\widetilde{B_3}$

where the antecedent $\widetilde{A_1}$, $\widetilde{A_2}$ and the consequent $\widetilde{B_1}$, $\widetilde{B_2}$ and $\widetilde{B_3}$ all are fuzzy sets. Once the inputs have been fuzzified, the level of satisfaction (membership value) of each part of the antecedent for each rule is known. The degree of satisfaction, *i.e.*, the membership value of the associated rule is called rule strength. If more than one antecedent are connected with "AND" operator then the *min* function is used to determine the rule strength and when they are connected with "OR" operator then the *max* function is used as follows:

$$\mu^{R_i} = \min\{\mu^{R_i}_{\tilde{A}_1}(x_1), \mu^{R_i}_{\tilde{A}_2}(x_2), \dots\}$$
$$\mu^{R_j} = \max\{\mu^{R_j}_{\tilde{A}_1}(x_1), \mu^{R_j}_{\tilde{A}_2}(x_2), \dots\}, \quad i, j \in \mathbb{N}$$

where $\mu^{R_i}_{\tilde{A}_1}(x_1), \mu^{R_i}_{\tilde{A}_2}(x_2), \dots$ are the membership values of the inputs $x_1, x_2, \dots$ to the antecedents $\widetilde{A_1}, \widetilde{A_2}, \dots$ of the rule R_i; corresponding $\mu^{R_j}_{\tilde{A}_1}(x_1), \mu^{R_j}_{\tilde{A}_2}(x_2), \dots$ for the rule R_j. Here, μ^{R_i} is the membership value when

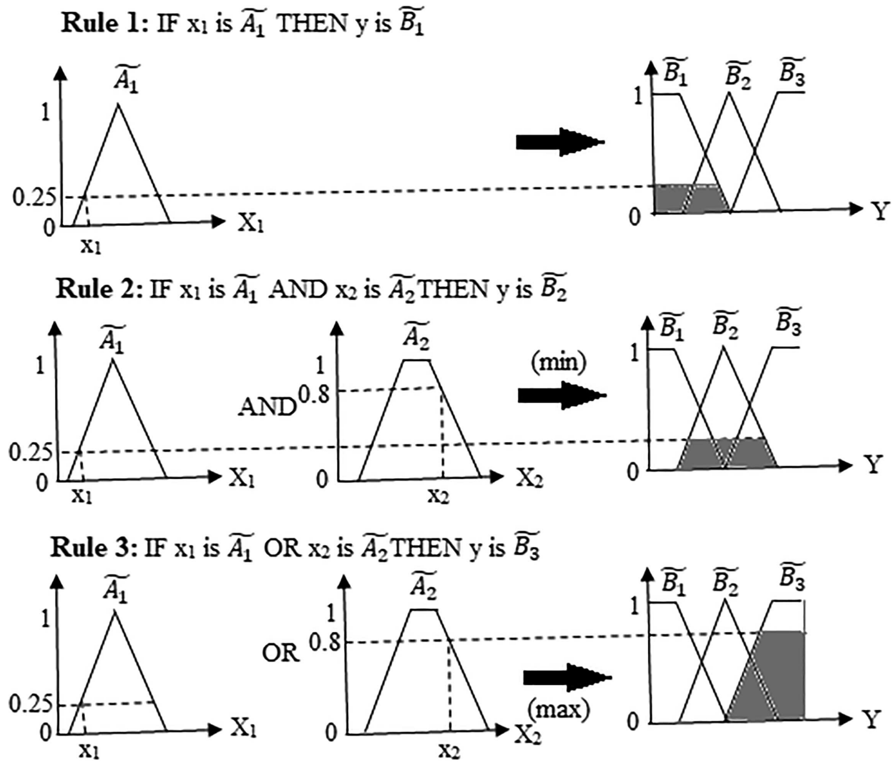


FIGURE D.2. Rule evaluation.

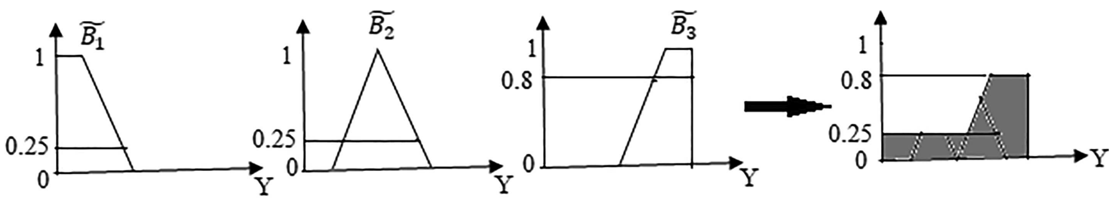


FIGURE D.3. Fuzzy output.

the antecedents are connected with “AND” operator in the i^{th} rule and μ^{R_j} is the membership value when they are connected with “OR” operator in the j^{th} rule Figure D.2.

(iii) **Fuzzy output:** After determining the rule strength of each rule, the region enclosed by the line that corresponds to the rule strength is the fuzzy output. It is calculated by combining them into a single fuzzy set with the aggregation operator $max\{\mu^{R_i}, \mu^{R_j}\}$, for all rules R_i and R_j Figure D.3.

(iv) **Defuzzification:** Defuzzification is the last step in this fuzzy inference process. Here, the combined fuzzy set obtained from the aggregation process yields a single crisp output value. There are several defuzzification techniques in the literature. Among them, centroid method (also named as center of gravity or center of area) is probably the most popular method. It is given by the formula:

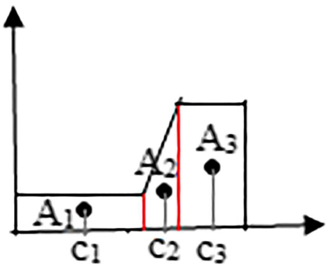


FIGURE D.4. Defuzzification through centroid method.

$$\text{COG}=\frac{\sum_{i=1}^{n} A_i c_i}{\sum_{i=1}^{n} A_i}$$

where A_i is the area and c_i is the centroid of the i^{th} sub-region Figure D.4.

(2) Sugeno method:

The Sugeno method which is also known as Takagi-Sugeno method was introduced by Takagi and Sugeno in 1985 [45]. This fuzzy inference technique is relatively same as the Mamdani method especially in the step of fuzzification of input values. The major difference is in the step of rule evaluation. In Mamdani method, both the antecedent and consequent are fuzzy sets, but here, instead of a fuzzy set, the consequent membership functions are mathematical functions of the input values which are either linear or constant. The IF-THEN type rule of the Sugeno method can be represented as follows:

Rule 1: IF x_1 is $\widetilde{A_1}$ THEN y is $f_1(x_1)$
Rule 2: IF x_1 is $\widetilde{A_1}$ AND x_2 is $\widetilde{A_2}$ THEN y is $f_2(x_1, x_2)$
Rule 3: IF x_1 is $\widetilde{A_1}$ OR x_2 is $\widetilde{A_2}$ THEN y is $f_3(x_1, x_2)$

where x_1 and x_2 are linguistic input values; $\widetilde{A_1}$ and $\widetilde{A_2}$ are fuzzy sets and $f_1(x_1)$, $f_2(x_1, x_2)$ and $f_3(x_1, x_2)$ are mathematical functions which are either linear or constant.

So, the typical rule of Sugeno method can be written as:

If 1^{st} Input $= x_1$ and 2^{nd} Input $= x_2$, then Output is $y = a_1x_1 + a_2x_2 + a_3$

where a_1, a_2 and a_3 are scalars.

For a zero-order Sugeno method, the output level y is a constant $(y = a_3, a_1 = a_2 = 0)$ and the rule can be written as:

IF x_1 is $\widetilde{A_1}$ AND x_2 is $\widetilde{A_2}$ THEN y is k

where k is a constant.

After the rule evaluation, rule strength is calculated similar as Mamdani method. Then a weighted-average mechanism is applied to determine the final crisp output as follows:

$$\text{y}=\frac{\sum_{i=1}^{n} w_i y_i}{\sum_{i=1}^{n} w_i}$$

where w_i is the rule strength and y_i is the value of the output y for the i^{th} rule. Sugeno method avoids the complicated defuzzification steps of the Mamdani method.

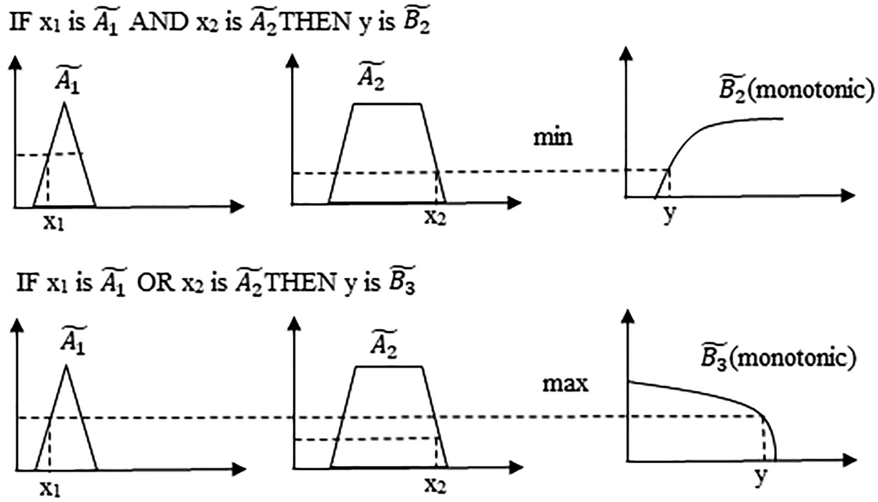


FIGURE D.5. Tsukamoto method.

(3) Tsukamoto method:

Tsukamoto method of fuzzy inference was first introduced by Tsukamoto in 1979 [47]. This method is a combination of Mamdani method and Sugeno method. The step of fuzzification of input values are same as the previously discussed two methods. The main difference in the step of rule evaluation. Similar as Mamdani method, here also the antecedent and consequent both are fuzzy sets but the only difference is that the consequent fuzzy set has a monotonic membership function. So, the IF-THEN type rule of the Tsukamoto method can be represented as follows:

Rule 1: IF x_1 is $\widetilde{A_1}$ THEN y is $\widetilde{B_1}$
Rule 2: IF x_1 is $\widetilde{A_1}$ AND x_2 is $\widetilde{A_2}$ THEN y is $\widetilde{B_2}$
Rule 3: IF x_1 is $\widetilde{A_1}$ OR x_2 is $\widetilde{A_2}$ THEN y is $\widetilde{B_3}$

where the consequent fuzzy sets $\widetilde{B_1}$, $\widetilde{B_2}$ and $\widetilde{B_3}$ have monotonic membership functions Figure D.5.

In the Tsukamoto method, the rule strength is calculated similar as Mamdani method and from the value of rule strength, the corresponding output value y is calculated for each rule. The next steps of getting final single crisp output is similar as Sugeno method, *i.e.*, by applying weighted-average mechanism. Here also, the complicated defuzzification step of Mamdani method is not required.

FUNDING

This work is supported by the Department of Science and Technology(DST), New Delhi, India, through the letter No.DST/INSPIRE/03/2016/001272.

CONFLICTS OF INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

AUTHOR CONTRIBUTION STATEMENT

Anushri Maji: Conceptualization, Methodology, Formal analysis, Writing - original draft, Visualization, Funding acquisition. **Asoke Kumar Bhunia:** Conceptualization, Methodology, Formal analysis, Visualization. **Shyamal Kumar Mondal:** Conceptualization, Methodology, Formal analysis, Visualization, Supervision.

Declaration of generative AI in scientific writing

During the preparation of this work, the authors used Grammarly and ChatGPT in order to improve readability and language. After using these tools, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

References

[1] A. Agrawal, D. Garg, R. Sethi and A.K. Shrivastava, Optimum redundancy allocation using spider monkey optimization. *Soft Comput.* **27** (2023) 15595–15608.

[2] N. Alikar, S.M. Mousavi, R.A.R. Ghazilla, M. Tavana and E.U. Olugu, Application of the nsga-ii algorithm to a multi-period inventory-redundancy allocation problem in a series-parallel system. *Reliab. Eng. Syst. Saf.* **160** (2017) 1–10.

[3] N. Alikar, S.M. Mousavi, R.A.R. Ghazilla, M. Tavana and E.U. Olugu, A bi-objective multi-period series-parallel inventory-redundancy allocation problem with time value of money and inflation considerations. *Comput. Ind. Eng.* **104** (2017) 51–67.

[4] M. Al-Salamah, Economic production quantity with the presence of imperfect quality and random machine breakdown and repair based on the artificial bee colony heuristic. *Appl. Math. Model.* **63** (2018) 68–83.

[5] M.A. Ardakan and A.Z. Hamadani, Reliability optimization of series–parallel systems with mixed redundancy strategy in subsystems. *Reliab. Eng. Syst. Saf.* **130** (2014) 132–139.

[6] M.A. Ardakan and A.Z. Hamadani, Reliability–redundancy allocation problem with cold-standby redundancy strategy. *Simul. Model. Pract. Theory* **42** (2014) 107–118.

[7] A. Banu and S.K. Mondal, An integrated inventory model with warranty dependent credit period under two policies of a manufacturer. *Opsearch* **55** (2018) 677–702.

[8] J.M. Betts, Minimizing inventory costs for capacity-constrained production using a hybrid simulation model. *Procedia Comput. Sci.* **29** (2014) 759–768.

[9] A.S. Bhandari, A. Kumar and M. Ram, Grey wolf optimizer and hybrid pso-gwo for reliability optimization and redundancy allocation problem. *Qual. Reliab. Eng. Int.* **39** (2023) 905–921.

[10] A.K. Bhunia, A.A. Shaikh and L.E. Cárdenas-Barrón, A partially integrated production-inventory model with interval valued inventory costs, variable demand and flexible reliability. *Appl. Soft Comput.* **55** (2017) 491–502.

[11] L.E. Cárdenas-Barrón, On optimal manufacturing batch size with rework process at single-stage production system. *Comput. Ind. Eng.* **53** (2007) 196–198.

[12] N. Chakraborty, S. Mondal and M. Maiti, A deteriorating multi-item inventory model with price discount and variable demands via fuzzy logic under resource constraints. *Comput. Ind. Eng.* **66** (2013) 976–987.

[13] A. Chambari, J. Sadeghi, F. Bakhtiari and R. Jahangard, A note on a reliability redundancy allocation problem using a tuned parameter genetic algorithm. *Opsearch* **53** (2016) 426–442.

[14] S. Chiu, Y. Huang, Y. Chiu and T. Chiu, Satisfying multiproduct demand with a fpr-based inventory system featuring expedited rate and scraps. *Int. J. Ind. Eng. Comput.* **10** (2019) 443–452.

[15] D.W. Coit, Cold-standby redundancy optimization for nonrepairable systems. *IIE Trans.* **33** (2001) 471–478.

[16] B.C. Das, B. Das and S.K. Mondal, Optimal transportation and business cycles in an integrated production-inventory model with a discrete credit period. *Transp. Res. E: Logist. Transp. Rev.* **68** (2014) 1–13.

[17] B.C. Das, B. Das and S.K. Mondal, An integrated production inventory model under interactive fuzzy credit period for deteriorating item with several markets. *Appl. Soft Comput.* **28** (2015) 453–465.

[18] K. Deb, Multi-objective Optimization Using Evolutionary Algorithms. John Wiley & Sons (2001).

[19] C. Franko, M. Ozkut and C. Kan, Reliability of coherent systems with a single cold standby component. *J. Comput. Appl. Math.* **281** (2015) 230–238.

[20] H. Garg, Fuzzy multiobjective reliability optimization problem of industrial systems using particle swarm optimization. *J. Ind. Math.* (2013).

[21] H. Garg and S.P. Sharma, Reliability-redundancy allocation problem of pharmaceutical plant. *J. Eng. Sci. Technol.* **8** (2013) 190–198.

[22] H. Gholinezhad, A new model for reliability redundancy allocation problem with component mixing. *Reliab. Eng. Syst. Saf.* **242** (2024) 109815.

[23] F. Grabski, 6-sm models of renewable cold standby system. In: *Semi-Markov Processes: Applications in System Reliability and Maintenance* (2015) 99–118.

[24] P.P. Guilani, P. Azimi, S.T.A. Niaki and S.A.A. Niaki, Redundancy allocation problem of a system with increasing failure rates of components based on weibull distribution: A simulation-based optimization approach. *Reliab. Eng. Syst. Saf.* **152** (2016) 187–196.

[25] W. Huang, J. Loman and T. Song, A reliability model of a warm standby configuration with two identical sets of units. *Reliab. Eng. Syst. Saf.* **133** (2015) 237–245.

[26] A.M.M. Jamal, B.R. Sarker, and S. Mondal, Optimal manufacturing batch size with rework process at a single-stage production system. *Comput. Ind. Eng.* **47** (2004) 77–89.

[27] H. Kim and P. Kim, Reliability–redundancy allocation problem considering optimal redundancy strategy using parallel genetic algorithm. *Reliab. Eng. Syst. Saf.* **159** (2017) 153–160.

[28] S.K. Mahato, L. Sahoo and A.K. Bhunia, Reliability-redundancy optimization problem with interval valued reliabilities of components via genetic algorithm. *J. Inf. Comput. Sci.* **7** (2012) 284–295.

[29] A. Maji, A.K. Bhunia and S.K. Mondal, Exploring a production-inventory model with optimal reliability of the production in a parallel-series system. *J. Ind. Prod. Eng.* **37** (2020) 71–86.

[30] A. Maji, A.K. Bhunia and S.K. Mondal, A production-reliability-inventory model for a series-parallel system with mixed strategy considering shortage, warranty period, credit period in crisp and stochastic sense. *Opsearch* **59** (2022) 862–907.

[31] E.H. Mamdani and S. Assilian, An experiment in linguistic synthesis with a fuzzy logic controller. *Int. J. Man-Machine Stud.* **7** (1975) 1–13.

[32] S. Mandal, K. Maity, S. Mondal and M. Maiti, Optimal production inventory policy for defective items with fuzzy time period. *Appl. Math. Model.* **34** (2010) 810–822.

[33] A.K. Manna, B. Das, J.K. Dey and S.K. Mondal, Multi-item epq model with learning effect on imperfect production over fuzzy-random planning horizon. *J. Manag. Anal.* **4** (2017) 80–110.

[34] A.K. Manna, J.K. Dey and S.K. Mondal, Imperfect production inventory model with production rate dependent defective rate and advertisement dependent demand. *Comput. Ind. Eng.* **104** (2017) 9–22.

[35] A.K. Manna, J.K. Dey and S.K. Mondal, Two layers supply chain in an imperfect production inventory model with two storage facilities under reliability consideration. *J. Ind. Prod. Eng.* **35** (2018) 57–73.

[36] A.K. Manna, J.K. Dey and S.K. Mondal, Controlling ghg emission from industrial waste perusal of production inventory model with fuzzy pollution parameters. *Int. J. Syst. Sci.: Oper. Logist.* **6** (2019) 368–393.

[37] S.M. Mousavi, V. Hajipour, S.T.A. Niaki and N. Alikar, Optimizing multi-item multi-period inventory control system with discounted cash flow and inflation: two calibrated meta-heuristic algorithms. *Appl. Math. Model.* **37** (2013) 2241–2256.

[38] S.M. Mousavi, N. Alikar, S.T.A. Niaki and A. Bahreininejad, Two tuned multi-objective meta-heuristic algorithms for solving a fuzzy multi-state redundancy allocation problem under discount strategies. *Appl. Math. Model.* **39** (2015) 6968–6989.

[39] A. Mukherjee and A.S. Dhar, Real-time fault-tolerance with hot-standby topology for conditional sum adder. *Microelectron. Reliab.* **55** (2015) 704–712.

[40] R. Nath and P.K. Muhuri, A novel evolutionary solution approach for many-objective reliability-redundancy allocation problem based on objective prioritization and constraint optimization. *Reliab. Eng. Syst. Saf.* **244** (2024) 109835.

[41] Z. Ouyang, Y. Liu, S.J. Ruan and T. Jiang, An improved particle swarm optimization algorithm for reliability-redundancy allocation problem with mixed redundancy strategy and heterogeneous components. *Reliab. Eng. Syst. Saf.* **181** (2019) 62–74.

[42] J. Sadeghi and S.T.A. Niaki, Two parameter tuned multi-objective evolutionary algorithms for a bi-objective vendor managed inventory model with trapezoidal fuzzy demand. *Appl. Soft Comput.* **30** (2015) 567–576.

[43] J. Sadeghi, S. Sadeghi and S.T.A. Niaki, A hybrid vendor managed inventory and redundancy allocation optimization problem in supply chain management: An nsga-ii with tuned parameters. *Comput. Oper. Res.* **41** (2014) 53–64.

[44] A. Salmasnia, E. Ameri and S.T.A. Niaki, A robust loss function approach for a multi-objective redundancy allocation problem. *Appl. Math. Model.* **40** (2016) 635–645.

[45] T. Takagi and M. Sugeno, Fuzzy identification of systems and its applications to modeling and control. *IEEE Trans. Syst. Man. Cybern.* (1985) 116–132.

[46] C.S. Tsou and C.H. Kao, Multi-objective inventory control using electromagnetism-like meta-heuristic. *Int. J. Prod. Res.* **46** (2008) 3859–3874.

[47] Y. Tsukamoto, An approach to fuzzy reasoning method. *Adv. Fuzzy Set Theory Appl.* (1979).

[48] C.E. Wells, Reliability analysis of a single warm-standby system subject to repairable and nonrepairable failures. *Eur. J. Oper. Res.* **235** (2014) 180–186.

[49] C. Yang, J. Gao and L. Sun, A multi-objective genetic algorithm for mixed-model assembly line rebalancing. *Comput. Ind. Eng.* **65** (2013) 109–116.

[50] T. Zhang and M. Horigome, Availability and reliability of system with dependent components and time-varying failure and repair rates. *IEEE Trans. Reliab.* **50** (2001) 151–158.

[51] Z. Zhang, L. Yang, Y. Xu, R. Zhu and Y. Cao, A novel reliability redundancy allocation problem formulation for complex systems. *Reliab. Eng. Syst. Saf.* **239** (2023) 109471.

[52] E. Zio and H. Gholinezhad, Redundancy allocation of components with time-dependent failure rates. *Mathematics* **11** (2023) 3534.