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EVALUATING DIRECTIONAL SCALE ELASTICITY IN THE RAM MODEL WITH UNDESIRABLE OUTPUTS: AN APPLICATION IN THE HORTICULTURAL INDUSTRY

Tong Xu[1], Reza Kazemi Matin[2,*] and Mahdi Mirjaberi[3]

Abstract. This paper advances the concept of scale elasticity by examining the impact of directional input changes on both desirable and undesirable outputs. We integrate two disposability concepts into a unified measure within the Range Adjusted Measure (RAM) model, enabling the computation of both directional scale elasticity and directional scale damage. This approach allows for a comprehensive analysis of scale effects in the presence of undesirable outputs. Grounded in Data Envelopment Analysis (DEA), our research enhances the understanding of managing undesirable outputs while accounting for directional scale changes, offering significant implications for both performance improvement and environmental sustainability. The paper includes theoretical foundations, numerical examples to illustrate the methodologies, and an empirical application in the Iranian horticulture industry, demonstrating the effectiveness and adaptability of our approach.

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1. Introduction

Data Envelopment Analysis (DEA) is a robust tool for evaluating productivity and performance across various sectors, including healthcare [24], education [7], horticulture [38], airlines [39], and manufacturing [52]. A critical focus of DEA research is the incorporation of undesirable outputs, which are essential for environmental protection and regulatory compliance [30, 31]. Managing these outputs is crucial for maintaining productivity and efficiency while mitigating environmental impact.

The literature extensively explores strategies for environmental improvement, including technological advancements, policy frameworks, and behavioral initiatives. Notable analyses, such as those on CO_2 emission performance across industrial sub-sectors in Shanghai, highlight the relationship between productivity growth and environmental sensitivity [15]. Research on CO_2 emissions quotas in Chinese manufacturing emphasizes the

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[1] Shangyu branch of Shaoxing University, Shaoxing City, Zhejiang Province, P.R. China.

[2] Department of Mathematics, College of Science, Sultan Qaboos University, Al-Khod 123, Muscat, Oman.

[3] Department of Mathematics, Faculty of Basic Sciences, Islamic Azad University, Esfahan (Khorasgan) Branch, Esfahan, Iran.

*Corresponding author: r.matin@squ.edu.om

importance of sustainable practices for achieving Net-Zero emissions [13, 14]. These contributions underscore the crucial link between productivity and sustainability.

In production economics, scale elasticity and scale efficiency are key metrics for understanding operational efficiency. Scale elasticity measures how productivity responds to changes in the scale of operations, providing insights into how output adjusts with varying input levels. Scale efficiency, expressed as a ratio, indicates whether a firm operates at its optimal size, which helps identify opportunities for enhancing productivity and reducing input costs.

Recent advancements in DEA models have improved the understanding of scale inefficiencies. The introduction of a three-stage computational procedure for DEA has allowed for the evaluation of radial efficiency, identification of efficient targets, and distinction of reference sets [41]. This method has been applied across various fields, including healthcare, education, environmental assessment, and manufacturing. Additionally, Forsund and Hjalmarsson [19] developed methods for computing scale elasticity, showing strong alignment between direct and indirect approaches. Krivonozhko *et al.* [26] proposed a parametric optimization algorithm for elasticity and scale effects, with notable applications in banking.

The concept of scale elasticity has evolved over time. Early work focused on the overall responsiveness of output to changes in inputs. Later advancements included directional elasticity, offering a geometric interpretation through parametric optimization algorithms [37]. The development of methods for estimating scale elasticity and returns to scale in non-radial DEA models further advanced the field [27]. Stochastic measures of scale elasticity were introduced, providing empirical findings from various sectors, including banking [1]. Further exploration included examining management preferences in directional scale elasticity for multi-output activities [46]. Innovations such as the range-adjusted measure (RAM) for returns to scale and non-radial measures of eco-efficiency have also contributed to a deeper understanding of how input adjustments affect both desirable and undesirable outputs [9, 33, 52].

Despite these advancements, no DEA-based studies have yet examined the effect of directional input changes on undesirable outputs, a concept we term "Directional Scale Damage" (DSD). This gap highlights the novelty of our approach. Our study introduces a measure to assess how changes in inputs impact undesirable outputs in any feasible direction, addressing limitations in previous methodologies that assumed full proportionality for scale measures and focused solely on natural strong disposability.

The key contributions of this study are:

- This study introduced Directional Scale Damage and extends the Range Adjusted Measure model to include both directional scale elasticity and directional scale damage within a unified framework, enabling a more nuanced analysis of efficiency.
- A comprehensive theoretical framework was developed, including efficient linear programming formulations for accurately computing the effects of directional input changes, thereby enhancing traditional DEA approaches.
- The approach was validated through an empirical study of 35 horticulture farms in Qazvin Province, Iran, demonstrating its effectiveness in integrating environmental and performance metrics for improved decision-making.

The paper is structured as follows: Section 2 reviews the literature on DEA, focusing on the modeling of undesirable factors and the concepts of scale efficiency and elasticity. Section 3 provides the necessary DEA background, explores efficiency from environmental and performance perspectives, enhances the RAM model, and discusses two existing scale measures and their computational processes. Section 4 presents the theoretical development and application of directional scale measures, including algebraic foundations and computational methods. Section 5 offers illustrative examples and an empirical application of the approach to 35 horticulture farms in Qazvin Province, Iran, demonstrating its effectiveness in integrating environmental and performance metrics. Section 6 concludes the paper and suggests directions for future research.

2. Literature review

2.1. Data Envelopment Analysis

DEA is a methodological approach that blends mathematical programming with multi-criteria decision-making principles. It's widely known for its effectiveness in evaluating and ranking Decision-Making Units (DMUs) based on multiple inputs and outputs. DEA has demonstrated its utility across various sectors such as agriculture, transportation, healthcare, finance, manufacturing, and education. Its capability to assess efficiency and performance across diverse contexts has made it a favored tool among researchers and practitioners alike. This is reflected in the extensive literature on DEA, encompassing numerous studies that delve into its applications and advancements. DEA offers several benefits, including its ability to operate without prior knowledge of production functions and constraints, manage multiple inputs and outputs simultaneously, accommodate diverse indicators of varying scales, compare inefficient DMUs directly with reference sets, rank DMUs, and provide benchmarks for inefficient units. However, DEA also has its limitations. It primarily focuses on relative efficiency rather than absolute efficiency, faces challenges in solving large-scale problems due to computational complexity, can be influenced by measurement errors that affect result accuracy, exhibits variability in performance evaluation based on changes in inputs and outputs, lacks the capability to conduct statistical tests like hypothesis testing due to its non-parametric nature, and its assessments can be sensitive to changes in sample composition [12, 29, 31, 32].

2.2. Modeling undesirable outputs

In traditional production economics, undesirable outputs often accompany desirable ones, leading to increased costs or other negative consequences. For instance, reducing undesirable outputs may require resources for clean-up efforts, production cutbacks, or fines. Moreover, the lack of prices for undesirable outputs complicates measuring their impact on overall productivity. Despite the joint production of good and bad outputs, traditional productivity analysis has largely overlooked undesirable outputs [8].

Färe *et al.* [17] proposed parametric methods to expand desirable outputs and contract undesirable ones. Hailu and Veeman [23] developed non-parametric productivity models that include undesirable outputs, presenting a non-orthodox monotonicity condition and arguing for the "weak disposability" implication in DEA. Färe and Grosskopf [16] contested this monotonicity condition, proposing a production technology with a single abatement parameter, aligning with Shephard's weak disposability of outputs, assuming a uniform abatement factor among firms.

Kuosmanen [28] introduced a non-uniform abatement factor for individual firms to define a true minimal extrapolation technology set. Pérez-López *et al.* [40] proposed a new method for calculating both time-invariant and time-variant scale inefficiency using DEA panel data in the presence of undesirable outputs. Walheer [51] offers the option to set different returns-to-scale assumptions and weak disposability for each desirable and undesirable output-specific production process and to choose between various types of convexity. Mahbubi *et al.* [34] proposed a linear programming approach based on DEA to compute the marginal rates of substitution for both single throughput and multi-throughput of efficient DMUs, considering the simultaneous presence of undesirable inputs and outputs.

DEA addresses the challenge of incorporating both desirable and undesirable outputs by providing a comprehensive productivity analysis that helps identify areas for improvement and benchmarks. It offers a robust tool for measuring and enhancing productivity across various settings. Given the global priority of environmental protection, firms must manage undesirable outputs to comply with regulations. This leads to two key strategies in the literature [48]:

- *Negative adaptation*: reducing inputs to decrease undesirable outputs while maximizing desirable outputs, subject to strong disposability.
- *Positive adaptation*: increasing inputs to boost desirable outputs while minimizing undesirable outputs, subject to managerial disposability.

While existing DEA methodologies have made significant progress in addressing undesirable outputs, there is still a need to refine these models by examining how changes in input levels specifically impact the production of undesirable outputs. This refinement could enhance the accuracy of environmental efficiency measures, improve the management of environmental impacts, and support better compliance with regulations, ultimately fostering greater sustainability.

2.3. Scale efficiency and scale elasticity

Scale efficiency is a pivotal metric in assessing productivity, defining the optimal size of operations for a unit. First introduced by Banker *et al.* [6] in performance analysis literature, it is expressed through the ratio of global efficiency, rooted in Constant Returns to Scale (CRS) technology, to local efficiency, associated with Variable Returns to Scale (VRS) technology. A scale-efficient unit operates at its optimal size, with global efficiency linked to CRS technology and local efficiency representing efficiency under VRS technology. Deviations from this optimal size adversely impact efficiency, highlighting the importance of scale efficiency in evaluating operational performance. Rödder *et al.* [47] propose a DEA-best practice criterion integrating efficiency and scaling effects, guiding DMUs toward optimal activity size and demonstrating practical benefits in Brazilian banks.

Another important measure is scale elasticity, or the total elasticity of production, which aggregates output elasticities to assess how outputs respond to marginal changes in inputs. Forsund [18] demonstrated that knowledge of scale efficiency does not suffice for calculating scale elasticity.

Podinovski [41] developed a three-stage computational procedure for DEA models to address scale inefficiencies in the standard radial method. This involves evaluating radial efficiency, identifying efficient targets, and distinguishing reference sets, each requiring one linear program per unit. These measures have been applied across various fields such as healthcare [24], higher education [7], environmental assessment [20–22, 31, 47, 48], and manufacturing [50, 52], underscoring the significance of scale efficiency and elasticity. Forsund and Hjalmarsson [19] proposed both direct and indirect methods for computing scale elasticity, finding strong correspondence between these methods when tested with real data. Krivonozhko *et al.* [26] developed a parametric optimization algorithm for computing elasticity and scale effects, demonstrating that analyzing a single interior point on a polyhedron face is sufficient. This approach has been applied to banking activities.

In addition to normative measures, returns to scale (RTS) describe the effectiveness of proportional changes in inputs on proportional changes in outputs. RTS attributes include constant RTS, decreasing RTS, and increasing RTS, as well as mixed attributes such as nonincreasing RTS, nondecreasing RTS, and variable RTS. These considerations may lead to strategic organizational changes, including mergers, acquisitions, and downsizing [42]. Podinovski *et al.* [44] derived a formula for scale elasticity in VRS technology without simplifying conditions. Podinovski and Forsund [43] used the directional derivative theorem to calculate marginal rates, extending elasticity measures to the efficient frontier using linear programs for VRS and other polyhedral technologies. Atici and Podinovski [4] extended this to CRS technologies, addressing undefined measures. Zelenxuk [52] explored scale elasticity using the directional distance function, establishing equivalence with input- and output-oriented measures and discussing DEA estimator-based estimation issues. Balk *et al.* [5] generalized the directional scale elasticity measure but focused on theoretical aspects, assuming a single direction without addressing computational aspects.

Cooper *et al.* [9] proposed the RAM model for RTS identification in non-radial DEA models, addressing multiple projections and supporting hyperplanes. Zelenyuk [53] also explored scale elasticity for directional distance functions, discussing duality with the profit function and DEA estimation issues. Lozano *et al.* [33] developed a Russell non-radial measure of eco-efficiency, applied to electric/electronic products, highlighting varying eco-efficiency returns to scale.

Miraberi and Kazemi Matin [37] extended the scale elasticity concept to evaluate elasticity in any direction, providing a geometric interpretation with a parametric optimization algorithm. Recent developments in non-radial DEA models have enhanced the understanding of efficiency and scale properties in various industries. Krivonozhko *et al.* [27] compared methods for estimating returns to scale in non-radial DEA models, emphasizing accurate calculations. Amirteimoori *et al.* [1] introduced a stochastic scale elasticity model addressing input proportions and price uncertainty, with empirical results from the Indian banking industry. Ren *et al.* [46] considered management preferences in directional scale elasticity for multi-output activities, demonstrated in Chinese research institutions.

Miraberi and Kazemi Matin [36] addressed scale elasticity with non-discretionary inputs using regression over efficiency measures to account for environmental effects. Goto and Sueyoshi [21,47] proposed a theoretical framework for measuring RTS under natural disposability and Damages to Scale under managerial disposability. However, their studies assumed full proportionality for both scale measures and that undesirable outputs met the requirement of natural strong disposability for RTS. This could be resolved by imposing natural weak disposability, as shown by Kuosmanen [28]. The study proposes a directional approach to address the first shortcoming, allowing scale measurement in any direction.

To fully capture environmental efficiency, it is crucial to extend the concept of scale elasticity by analyzing how directional input changes impact the production of undesirable outputs. Current methods focus on overall output responses but often overlook the specific effects on undesirable byproducts. By incorporating this dimension, we can gain a clearer understanding of how input adjustments influence both desirable and undesirable outcomes, leading to more effective environmental management and sustainability strategies.

3. Background

This section briefly reviews requisites for the rest of paper. Here, we generally assume that the input quantities are represented by vector $\mathbf{x} = (x_1, \ldots, x_N)^t \in R_+^N$, the desirable or good outputs by $\mathbf{y} = (y_1, \ldots, y_M)^t \in R_+^M$, and the undesirable or bad outputs by $\mathbf{z} = (z_1, \ldots, z_K)^t \in R_+^K$, where the superscript "t" denotes the transpose vector. The production technology is characterized by production set $\mathcal{T} = \{(\mathbf{x}, \mathbf{y}, \mathbf{z})\colon \mathbf{x} \text{ can produce } (\mathbf{y}, \mathbf{z})\}$, or alternatively, by output set $\mathcal{P}(\mathbf{x}) = \{(\mathbf{y}, \mathbf{z}) : (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathcal{T}\}, \mathbf{x} \in R_+^N$.

Consider a set of J observations of actual production units $(\mathbf{x}_j, \mathbf{y}_j, \mathbf{z}_j),\ j = 1, \ldots, J$. Following Kuosmanen [25], under the assumptions of free disposability of inputs and good outputs, weak disposability of both good and bad outputs, convexity of $\mathcal{T}$, and variable returns to scale, the empirical output set exhibiting weak disposability is given by:

$$\mathcal{P}_W(\mathbf{x}) = \left\{ (\mathbf{y}, \mathbf{z}) : \begin{array}{ll} \displaystyle\sum_{j=1}^{J} x_{nj}(\lambda_j + \mu_j) \leq x_n, & n = 1, \ldots, N, \\ \displaystyle\sum_{j=1}^{J} y_{mj}\lambda_j \geq y_m, & m = 1, \ldots, M, \\ \displaystyle\sum_{j=1}^{J} z_{kj}\lambda_j = z_k, & k = 1, \ldots, K, \\ \displaystyle\sum_{j=1}^{J} (\lambda_j + \mu_j) = 1, & \\ \lambda_j, \mu_j \geq 0, & j = 1, \ldots, J \end{array} \right\}. \tag{1}$$

This formulation employs the variables λ_j and μ_j as intensity weights for constructing convex combinations of observed production units (designated as j), grounded in the convexity assumption. The enforcement of variable returns to scale is maintained by constraining the intensity variables to collectively sum to 1. The demonstration of free disposability for inputs and desirable outputs incorporates inequality constraints, while equality constraints are utilized for undesirable outputs to reflect the weak disposability assumption.

Importantly, the technology described in this context demonstrates linearity concerning the unknown intensity variables λ_j and μ_j. Implementing the weak disposability assumption for outputs, the desirable and undesirable outputs of production unit j are respectively weighted by the non-disposed portion λ_j, while the inputs are weighted by the sum of the disposed and non-disposed components. The variable λ_j interprets as the portion of production unit j's output that remains active, whereas μ_j denotes the portion of output that is abated through the scaling down of activity levels, as explained by Kuosmanen [25].

Following Goto and Sueyoshi [20], under free disposability of inputs, good outputs, and bad outputs, variable returns to scale, and convexity, the empirical output set, exhibiting managerial disposability, is given by:

$$\begin{aligned}\mathcal{P}_M(\mathbf{x}) = \Bigg\{(\mathbf{y},\mathbf{z}) :\ & \sum_{j=1}^{J} x_{nj}\eta_j \geq x_n, && n = 1,\ldots,N,\\ & \sum_{j=1}^{J} y_{mj}\eta_j \geq y_m, && m = 1,\ldots,M,\\ & \sum_{j=1}^{J} z_{kj}\eta_j \leq z_k, && k = 1,\ldots,K,\\ & \sum_{j=1}^{J} \eta_j = 1,\ \eta_j \geq 0, && j = 1,\ldots,J\Bigg\}.\end{aligned} \tag{2}$$

In this formulation, the variables η_j serve as intensity weights. The level of inefficiency here is compared to the efficiency frontier of this unified output set, leading to a more robust and realistic assessment of efficiency.

3.1. Extending the RAM model for a unified operational and environmental evaluation

The concepts of operational and environmental assessments of firms arise in the literature from a choice of either negative or positive adaptation. Similar to Goto and Sueyoshi [19], we modify the RAM model to measure operational and environmental performances simultaneously, though our modification differs from theirs to some extent. Recent studies investigating RAM have introduced significant advancements. For example, Aparicio [3] presented a comprehensive model incorporating new computational features. These RAM models have proven valuable in various environmental applications, offering robust evaluation tools and enhanced analytical capabilities [11, 25].

The RAM in DEA offers a nuanced approach to assessing the efficiency of DMUs. By incorporating the concept of range adjustment, RAM provides a way to handle data variability more effectively than traditional DEA models. This adjustment allows for a more balanced comparison of DMUs by accounting for differences in the scale and range of input-output data. Consequently, RAM helps identify truly efficient units by mitigating the influence of outliers and extreme values, leading to more accurate and reliable efficiency scores [9].

Additionally, RAM enhances the discriminatory power of DEA models. Traditional DEA approaches sometimes classify multiple units as efficient, particularly when dealing with a large number of DMUs or a limited number of inputs and outputs. RAM addresses this limitation by adjusting the efficiency scores based on the relative ranges of data, thereby providing a finer granularity of performance assessment. This improved discrimination aids decision-makers in identifying best practices and areas for improvement, fostering a deeper understanding of operational efficiencies and inefficiencies within the evaluated entities [9].

The unified performance measure for DMU$_o$ in the RAM model can be set up as follows:

$$\begin{aligned}\text{Maximize}\ & \sum_{n=1}^{N} r_n^i\big(s_n^{i-} + s_n^{i+}\big) + \sum_{m=1}^{M} r_m^g s_m^g + \sum_{k=1}^{K} r_k^b s_k^b\\ \text{Subject to}\ & \sum_{j=1}^{J} x_{nj}(\lambda_j + \mu_j) + s_n^{i-} = x_{no}, \qquad n = 1,\ldots,N,\end{aligned}$$

$$
\begin{aligned}
&\sum_{j=1}^{J} y_{mj}\lambda_j - s_m^g = y_{mo}, && m = 1, \ldots, M,\\
&\sum_{j=1}^{J} z_{kj}\lambda_j = z_{ko}, && k =, \ldots, K,\\
&\sum_{j=1}^{J} (\lambda_j + \mu_j) = 1 &&\\
&\sum_{j=1}^{J} x_{nj}\eta_j - s_n^{i+} = x_{no}, && n = 1, \ldots, N,\\
&\sum_{j=1}^{J} y_{mj}\eta_j \geq y_{mo}, && m = 1, \ldots, M,\\
&\sum_{j=1}^{J} z_{kj}\eta_j + s_k^b = z_{ko}, && k = 1, \ldots, K,\\
&\sum_{j=1}^{J} \eta_j = 1 &&\\
&\lambda_j, \mu_j, \eta_j \geq 0, && j = 1, \ldots, J,\\
&s_n^{i-}, s_n^{i+} \geq 0, && n = 1, \ldots, N\\
&s_k^b \geq 0, && k = 1, \ldots, K,\\
&s_m^g \geq 0, && m = 1, \ldots, M
\end{aligned}
\tag{3}
$$

where $s_n^{i\mp}$ $(n = 1, \ldots, N)$, s_m^g $(m = 1, \ldots, M)$, and s_k^b $(k = 1, \ldots, K)$ are all slack variables related to inputs, desirable (good) outputs, and undesirable (bad) outputs, respectively. The slack variables $s_n^{i\mp}$, incorporated respectively in the first and third group of constraints, related to inputs according to their negative and positive adaptations. Vectors $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_J)^t$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_J)^t$, and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_J)^t$ represent unknown intensity variables employed to form combinations of actual observations.

The range in model (3) is specified as: $r_n^i = 1/[(M+N+K)(\overline{x}_n - \underline{x}_n)]$ for $n = 1, \ldots, N$, $r_m^g = 1/[(M+N+K)(\overline{y}_m - \underline{y}_m)]$ for $m = 1, \ldots, M$, and $r_k^b = 1/[(M+N+K)(\overline{z}_k - \underline{z}_k)]$ for $k = 1, \ldots, K$ where $\overline{x}_n = \max_j\{x_{nj}\}$ and $\underline{x}_n = \min_j\{x_{nj}\}$, and the others are likewise. Model (3) detects both the environmental and operational inefficiencies simultaneously, therefore a unified efficiency measure (UEM) on its optimality is given by:

$$
\mathcal{E}_o = 1 - \left[\sum_{n=1}^{N} r_n^i \left(s_n^{i-^*} + s_n^{i+^*}\right) + \sum_{m=1}^{M} r_m^g s_m^{g^*} + \sum_{k=1}^{K} r_k^b s_k^{b^*}\right]. \tag{4}
$$

In model (3), the constraints $\sum_{j=1}^{J} x_{nj}\lambda_j \leq x_{no}$ $(n = 1, \ldots, N)$ and $\sum_{j=1}^{J} y_{mj}\lambda_j \geq y_{mo}$ $(m = 1, \ldots, M)$ indicates that the efficiency frontier for good outputs locates on or above the convex combinations of all observations. Meanwhile, the constraints $\sum_{j=1}^{J} x_{nj}\eta_j \geq x_{no}$ $(n = 1, \ldots, N)$ and $\sum_{j=1}^{J} z_{kj}\eta_j \leq z_{ko}$ $(k = 1, \ldots, K)$ indicates that the efficiency frontier for bad outputs locates on or below the convex combinations of all observations.

Here, the unit o under assessment is *fully efficient* if it is both environmental and operational efficient, that is, unit o is fully efficient if and only if $\mathcal{E}_o = 1$. Otherwise, the unit is *partially* efficient (either operational or environmental efficient and nor both) or inefficient.

– For an inefficient unit o, it needs to improve its operational performance by decreasing inputs *via* $\mathbf{s}^{i-^*}$ and increasing desirable outputs *via* $\mathbf{s}^{g^*}$. Of course, decreasing inputs always yield to *natural* reduction of undesirable outputs.

– On the other hand, unit o can improve its environmental performance by increasing inputs *via* $\mathbf{s}^{i+^*}$ and decreasing undesirable outputs *via* $\mathbf{s}^{b^*}$. This approach concerns the *managerial* effort for environment protection.

To show the fesibility of this model, let $\lambda_o = \mu_o = \eta_o = 1$, $\lambda_j = \mu_j = \eta_j = 0$ $(\forall j \neq o)$, also $s_n^{i-} = s_n^{i+} = 0$ $(\forall n)$, $s_k^b = 0$ $(\forall k)$ and $s_m^g = 0$ $(\forall m)$ which satisfy all the constraints of the model (3).

Choosing RAM, given its linear structure, allows us to incorporate inputs and both desirable and undesirable values in efficiency measurement. RAM's unique capabilities, such as providing a strong efficient target point and facilitating dual formulations, including the structured application of supporting hyperplanes, were instrumental for our study. These features enable a more comprehensive and nuanced assessment of efficiency, accommodating a wider range of operational and environmental factors.

The dual formulation for model (3) has a mathematical structure as:

$$\begin{aligned}
\text{Minimize}\ & \sum_{n=1}^{N}(v_n^1 - v_n^2)x_{no} - \sum_{m=1}^{M}(u_m^1 + u_m^2)y_{mo} + \sum_{k=1}^{K}(w_k^1 + w_k^2)z_{ko} + u_0^1 + u_0^2 & \\
\text{Subject to}\ & \sum_{n=1}^{N} v_n^1 x_{nj} - \sum_{m=1}^{M} u_m^1 y_{mj} + \sum_{k=1}^{K} w_k^1 z_{kj} + u_0^1 \geq 0, & j = 1, \ldots, J, \\
& \sum_{n=1}^{N} v_n^1 x_{nj} + u_0^1 \geq 0, & j = 1, \ldots, J, \\
& -\sum_{n=1}^{N} v_n^2 x_{nj} - \sum_{m=1}^{M} u_m^2 y_{mj} + \sum_{k=1}^{K} w_k^2 z_{kj} + u_0^2 \geq 0, & j = 1, \ldots, J, \\
& v_n^1, v_n^2 \geq 0, & n = 1, \ldots, N, \\
& u_m \geq 0, & m = 1, \ldots, M, \\
& w_k^2 \geq 0, & k = 1, \ldots, K, \\
& w_k^1, u_0^1, u_0^2 \text{ unrestricted}, & k = 1, \ldots, K.
\end{aligned} \tag{5}$$

This model represents a novel linear programming framework designed to measure the efficiency of DMUs by simultaneously considering inputs, desirable outputs, and undesirable outputs. The objective function aims to minimize the weighted sum of inputs and undesirable outputs while maximizing the weighted sum of desirable outputs.

Two key features of the model is as follows:

– *Operational Frontier* (Usual DEA Frontier): the first set of constraints forms the operational frontier, which is constructed using supporting hyperplanes. This frontier represents the maximum feasible output for given inputs under the assumption of strong disposability of inputs and desirable outputs. It defines the efficient production possibility set where the DMUs should ideally operate.
– *Environmental Frontier*: the second set of constraints accommodates the weak disposability of undesirable outputs. This frontier adjusts for the presence of undesirable outputs and ensures that reducing these outputs does not compromise the efficiency assessment.

As the result, the supporting hyperplane for unit o, the hyperplane onto which it is projected operationally at optimality of model (5), is given by:

$$\sum_{n=1}^{N} v_n^{1^*}\Bigg(\underbrace{x_{no} - s_n^{i-^*}}_{x_{no}^{-^*}}\Bigg) - \sum_{m=1}^{M} u_m^{1^*}\Bigg(\underbrace{y_{mo} + s_m^{g^*}}_{y_{mo}^{-^*}}\Bigg) + \sum_{k=1}^{K} w_k^{1^*} z_{ko}^{-^*} + u_0^{1^*} = 0. \tag{6}$$

Secondly, the environmental frontier is set up by supporting hyperplanes provided by the third group of constraints in model (5). The supporting hyperplane for unit o, the hyperplane onto which it is projected environmentally, is given by:

$$\sum_{n=1}^{N} v_n^{2^*}\left(\underbrace{x_{no}+s_n^{i+^*}}_{x_{no}^{+^*}}\right)+\sum_{m=1}^{M} u_m^{2^*} y_{mo}^{+^*}-\sum_{k=1}^{K} w_k^{*}\left(\underbrace{z_{ko}-s_k^{b^*}}_{z_{ko}^{+^*}}\right)-u_0^{2^*}=0. \tag{7}$$

The use of supporting hyperplanes provides a robust efficient target point, offering a clear benchmark for performance improvement. This approach allows for a structured and systematic application, which is crucial for developing our methodology in evaluating DTS for efficient (or projected) units.

3.2. Measuring scale elasticity and scale damage using the extended RAM model

Scale elasticity is crucial in performance analysis as it guides managers in making informed decisions about the contraction or expansion of firm operations. In essence, SE describes how outputs respond to changes in inputs, providing insight into the scalability of operations and the potential for efficiency improvements.

To calculate scale elasticity for DMU_o, consider its projection onto the efficient operational frontier using the optimal solution of the extended RAM model (3). This is expressed as: $\mathbf{x}_o^{-^*}=\mathbf{x}_o-\mathbf{s}^{i-^*}$, $\mathbf{y}_o^{-^*}=\mathbf{y}_o+\mathbf{s}^{g^*}$, and $\mathbf{z}_o^{-^*}=\sum_{j=1}^{J}\lambda_j^*\mathbf{z}_j$. Forsund [12] asserts that:

$$\begin{aligned}
\mathrm{SE}_o&=\frac{\sum_{n=1}^{N} v_n^{1^*} x_{no}^{-^*}}{\sum_{m=1}^{M} u_m^{1^*} y_{mo}^{-^*}}=\frac{\left(\sum_{m=1}^{M} u_m^{*} y_{mo}^{-^*}-\sum_{k=1}^{K} w_k^{1^*} z_{ko}^{-^*}-u_0^{1^*}\right)}{\sum_{m=1}^{M} u_m^{1^*} y_{mo}^{-^*}} \qquad \text{[From 6]}\\
&=1-\left(\sum_{k=1}^{K} w_k^{1^*} z_{ko}^{-^*}+u_0^{1^*}\right)\Big/\sum_{m=1}^{M} u_m^{1^*} y_{mo}^{-^*}.
\end{aligned} \tag{8}$$

From (8) it follows that scale elasticity depends on variation of $\sum_{k=1}^{K} w_k^{1^*} z_{ko}^{-^*}+u_0^{1^*}$ provided that the denominator is set to unity. Note that the expression in denominator is positive for at least there is a m so that $u_m^{1^*} y_{mo}^{-^*}>0$. So, the range of scale elasticity is simply calculated by solving a pair of linear programming, as follows:

$$\begin{aligned}
\text{Maximize/Minimize } & \sum_{k=1}^{K} w_k^1 z_{ko}^{-^*}+u_0^1\\
\text{Subject to } & \sum_{m=1}^{M} u_m^1 y_{mo}^{-^*}=1\\
& \sum_{n=1}^{N} v_n^1 x_{no}^{-^*}-\sum_{m=1}^{M} u_m^1 y_{mo}^{-*}+\sum_{k=1}^{K} w_k^1 z_{ko}^{-^*}+u_0^1=0\\
& \sum_{n=1}^{N} v_n^1 x_{nj}-\sum_{m=1}^{M} u_m^1 y_{mj}+\sum_{k=1}^{K} w_k^1 z_{kj}+u_0^1\geq 0, && j=1,\ldots,J\\
& \sum_{n=1}^{N} v_n^1 x_{nj}+u_0^1\geq 0, && j=1,\ldots,J\\
& v_n^1\geq 0, && n=1,\ldots,N\\
& u_m^1\geq 0, && m=1,\ldots,M\\
& w_k^1, u_0^1 \text{ unrestricted}, && k=1,\ldots,K.
\end{aligned} \tag{9}$$

Remark 1. Note that, due to the feasibility of the linear model (3) and the duality relations, the introduced model (9) is feasible. Furthermore, considering the first two constraints of this model, we have: $\sum_{k=1}^{K} w_k^1 z_{ko}^{-^*} + u_0^1 = \sum_{m=1}^{M} u_m^1 y_{mo}^{-^*} - \sum_{n=1}^{N} v_n^1 x_{no}^{-^*} = 1 - \sum_{n=1}^{N} v_n^1 x_{no}^{-^*} \leq 1$. This indicates that the objective value in model (9) is bounded above. However, when calculating the range of scale elasticity (SE_o), the minimum value can be unbounded.

Goto and Sueyoshi [19] introduced a new scale measure, termed "scale damage" (SD) in this study, to address the presence of undesirable outputs. This measure captures the impact of input changes on undesirable outputs within the environmental frontier.

Assuming the projection of DMU_o onto the efficient environmental frontier, based on the optimal solution of the extended RAM model (3), is given by: $\mathbf{x}_o^{+^*} = \mathbf{x}_o + \mathbf{s}^{i+^*}$, $\mathbf{y}_o^{+^*} = \sum_{j=1}^{J} \eta_j^* \mathbf{y}_j$, and $\mathbf{z}_o^{+^*} = \mathbf{z}_o - \mathbf{s}^{b^*}$. Then the SD value is expressed as:

$$\begin{aligned}\mathrm{SD}_o &= \frac{\sum_{n=1}^{N} v_n^{2^*} x_{no}^{+^*}}{\sum_{k=1}^{K} w_k^{2^*} z_{ko}^{+^*}} = \left(-\sum_{m=1}^{M} u_m^{2^*} y_{mo}^{+^*} + \sum_{k=1}^{K} w_k^* z_{ko}^{+^*} + u_0^{2^*} \right) / \sum_{k=1}^{K} w_k^{2^*} z_{ko}^{+^*} \qquad \text{[From 7]} \\ &= 1 - \left(\sum_{m=1}^{M} u_m^{2^*} y_{mo}^{+^*} - u_0^{2^*} \right) / \sum_{k=1}^{K} w_k^{2^*} z_{ko}^{+^*} n. \end{aligned} \tag{10}$$

From (10) it follows that scale damage depends on variation of $\sum_{m=1}^{M} u_m^{2^*} y_{mo}^{+^*} - u_0^{2^*}$ provided that the denominator is set to unity. Note that the expression in denominator is positive because at least there is a k so that $w_k^{2^*} z_{ko}^* > 0$. Thus, the range of scale damages is simply calculated by solving a pair of linear programming, as follows:

$$\begin{aligned} \text{Maximize/Minimize } & \sum_{m=1}^{M} u_m^2 y_{mo}^{+^*} - u_0^2 \\ \text{Subject to } & \sum_{k=1}^{K} w_k^2 z_{ko}^{+^*} = 1 \\ & \sum_{n=1}^{N} v_n^2 x_{no}^{+^*} + \sum_{m=1}^{M} u_m^2 y_{mo}^{+^*} - \sum_{k=1}^{K} w_k^2 z_{ko}^{+^*} - u_0^2 = 0 \\ & \sum_{n=1}^{N} v_n^2 x_{nj} + \sum_{m=1}^{M} u_m^2 y_{mj} - \sum_{k=1}^{K} w_k^2 z_{kj} - u_0^2 \leq 0, && j = 1, \ldots, J \\ & v_n^2 \geq 0, && n = 1, \ldots, N \\ & u_m^2 \geq 0, && m = 1, \ldots, M \\ & w_k^2 \geq 0, && k = 1, \ldots, K \\ & u_0^2 \text{ unrestricted.} \end{aligned} \tag{11}$$

Remark 2. Note that, again, due to the feasibility of the linear model (3) and the duality relations, model (11) is always feasible. Furthermore, considering the first two constraints of this model, we have: $\sum_{m=1}^{M} u_m^2 y_{mo}^{+^*} - u_0^2 \leq 1$. This indicates that the objective value in model (11) is bounded above. However, the minimum value can be unbounded. Based on relation (10), when calculating the range of scale damage (SD_o), the upper bound can be $+\infty$.

4. Directional scale measure

This section further extends the approach for evaluating elasticity measures, introduced in the previous section using the extended RAM model, to more general concepts, specifically the directional scale damage measure. This advancement allows for the assessment of how changes in inputs impact undesirable outputs in any feasible direction. It addresses limitations in previous methodologies that assumed full proportionality for scale measures and focused solely on natural strong disposability.

We herewith assume that vector of inputs are interrupted in a direction $\mathbf{d}^i \in R_+^N$ and investigate its influence over good outputs in a direction $\mathbf{d}^g \in R_+^M$ and bad outputs in a direction $\mathbf{d}^b \in R_+^K$.

Supposing vector of inputs are interrupted in the given direction by α, then vector of good and bad outputs should be respectively interrupted in the given directions by $\beta(\alpha)$ and $\gamma(\alpha)$ so that points $(\mathbf{x}_o^{-*} + \alpha\mathbf{d}^i, \mathbf{y}_o^* + \beta(\alpha)\mathbf{d}^g, \mathbf{z}_o)$ and $(\mathbf{x}_o^{+^*} + \alpha\mathbf{d}^i, \mathbf{y}_o, \mathbf{z}_o + \gamma(\alpha)\mathbf{d}^b)$ remain on the frontier. Using Lagrangian function on optimal solution of model (3) and differentiate it at $\alpha = 0$ gives the *directional scale elasticity* (DSE) and *directional scale damage* (DSD), denoted respectively by $\mathrm{SE}_o(\mathbf{d})$ and $\mathrm{SD}_o(\mathbf{d})$, as:

$$\mathrm{SE}_o(\mathbf{d}) := \beta'(0) = \sum_{n=1}^{N} v_n^{1^*} d_n^i \Big/ \sum_{m=1}^{M} u_m^{1^*} d_m^g \tag{12}$$

$$\mathrm{SD}_o(\mathbf{d}) := \gamma'(0) = \sum_{n=1}^{N} v_n^{2^*} d_n^i \Big/ \sum_{k=1}^{K} w_k^{2^*} d_k^b. \tag{13}$$

Comparing equations (12) with (8) and (13) with (10), it is found that if we set $\mathbf{d}^i = \mathbf{x}_o$, $\mathbf{d}^g = \mathbf{y}_o$, and $\mathbf{d}^b = \mathbf{z}_o$, then directional scale measures are identical to their corresponding conventional scale measures. DSE has already been addressed by Mirjaberi and Kazemi Matin [36], this paper focuses on developing directional scale damage.

To obtain optimal dual variables in (13) we can solve model (5) or directly solve the following fractional programming:

$$\begin{aligned}
\text{Maximize/Minimize}\ & \sum_{n=1}^{N} v_n^2 d_n^i \Big/ \sum_{k=1}^{K} w_k^2 d_k^b \\
\text{Subject to}\ & \sum_{n=1}^{N} v_n^2 x_{no}^{+^*} + \sum_{m=1}^{M} u_m^2 y_{mo}^{+^*} - \sum_{k=1}^{K} w_k^2 z_{ko}^{+^*} - u_0^2 = 0 \\
& \sum_{n=1}^{N} v_n^2 x_{nj} + \sum_{m=1}^{M} u_m^2 y_{mj} - \sum_{k=1}^{K} w_k^2 z_{kj} - u_0^2 \le 0, && j = 1, \ldots, J \\
& v_n^2 \ge 0, && n = 1, \ldots, N \\
& u_m^2 \ge 0, && m = 1, \ldots, M \\
& w_k^2 \ge 0, && k = 1, \ldots, K \\
& u_0^2 \text{ unrestricted.}
\end{aligned} \tag{14}$$

Using Charnes and Cooper's transformation, model (14) is turned into the following linear programming:

$$\begin{aligned}
\text{Maximize/Minimize}\ & \sum_{n=1}^{N} v_n^2 d_n^i \\
\text{Subject to}\ & \sum_{k=1}^{K} w_k^2 d_k^b = 1, \\
& \sum_{n=1}^{N} v_n^2 x_{no}^{+^*} + \sum_{m=1}^{M} u_m^2 y_{mo}^{+^*} - \sum_{k=1}^{K} w_k^2 z_{ko}^{+^*} - u_0^2 = 0,
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=1}^{N} v_n^2 x_{nj} + \sum_{m=1}^{M} u_m^2 y_{mj} - \sum_{k=1}^{K} w_k^2 z_{kj} - u_0^2 \leq 0, && j = 1, \dots, J\\
&v_n^2 \geq 0, && n = 1, \dots, N\\
&u_m^2 \geq 0, && m = 1, \dots, M\\
&w_k^2 \geq 0, && k = 1, \dots, K\\
&u_0^2 \text{ unrestricted.} && (15)
\end{aligned}$$

Model (15) distinguishes lower and upper bounds for directional scale damage, thereupon, if the point under study is a relative interior point on the outer facet of environmental frontier, the bounds coincide. This is referred to as differentiable case in the literature.

4.1. Discussion and analysis

This section briefly discusses the algebra behind the directional scale damage. For any point $(\mathbf{x}, \mathbf{z})^t$ on the outer facet of environmental frontier, the function $f(\mathbf{x}, \mathbf{z}) = 0$ in the space of R^{N+K} that characterizes the frontier is implicitly given by equations as (7). Supposing the plane $Pl(\mathbf{x}, \mathbf{z}; \mathbf{d}) := \{(\mathbf{x} + \alpha \mathbf{d}^i, \mathbf{z} + \gamma \mathbf{d}^b) : \alpha, \gamma \in R\}$ intersects f at $(\mathbf{x}_o^{+^*}, \mathbf{z}_o^*)$, a relative interior point on the outer facet of environmental frontier. This gives the curve C as:

$$\alpha \sum_{n=1}^{N} v_n^{2^*} d_n^i - \gamma \sum_{k=1}^{K} w_k^* d_k^b = 0$$

which implies:

$$\gamma = \alpha \sum_{n=1}^{N} v_n^{2^*} d_n^i / \sum_{k=1}^{K} w_k^* d_k^b$$

provided that the denominator is non-zero. Since the target point is a relative interior point, γ is differentiable at $\alpha = 0$, otherwise one-sided derivatives exist[1]. Differentiating the latter expression at $\alpha = 0$ gives equation (13), so directional scale damage is geometrically the tangent to the curve C at $(\mathbf{x}_o, \mathbf{z}_o)^t$.

To deduce equation (13) algebraically, let $(\mathbf{y}, \mathbf{z})^t \in \mathcal{P}_M(\mathbf{x})$ and for any $\mathbf{d}^i \neq \mathbf{0}_N$, define:

$$\mathcal{A} := \{\alpha \in R : \mathcal{P}_M(\mathbf{x} + \alpha \mathbf{d}^i) \neq \emptyset\}.$$

Note that $\mathcal{A} \neq \emptyset$ for $\mathbf{0} \in \mathcal{A}$. If $\mathcal{A}$ is not the singleton set $\{\mathbf{0}\}$, convexity of $\mathcal{P}_M(\mathbf{x})$ implies that $\mathcal{A}$ is a convex subset of real line. Assuming $\mathcal{A} \neq \{\mathbf{0}\}$, then for any $\mathbf{d}^b \neq \mathbf{0}_K$ the directional response function $\gamma : \mathcal{A} \mapsto [-\infty, +\infty]$ is defined by

$$\gamma(\alpha) := \max\{\sigma : (\mathbf{y}, \mathbf{z} + \sigma \mathbf{d}^b)^t \in \mathcal{P}_M(\mathbf{x} + \alpha \mathbf{d}^i)\}. \tag{16}$$

In terms, $\gamma(\alpha)$ is the maximum adjustment of output vector $(\mathbf{y}, \mathbf{z})^t$ on the environmental frontier along direction $(\mathbf{0}_M, \mathbf{d}^b)^t$ allowed in the perturbed output set $\mathcal{P}_M(\mathbf{x} + \alpha \mathbf{d}^i)$.

Lemma 1 ([16]). *Supposing $\mathcal{T}$ satisfies managerial disposability of input and outputs, convexity, and variable returns to scale. Then, for any $\mathbf{x}_1, \mathbf{x}_2 \in R_+^N$ we have:*

(P1) *$\mathbf{x}_1 \leq \mathbf{x}_2$ implies that $\mathcal{P}_M(\mathbf{x}_1) \supseteq \mathcal{P}_M(\mathbf{x}_2)$.*
(P2) *If $(\mathbf{y}, \mathbf{z})_1 \in \mathcal{P}_M(\mathbf{x}^1)$, $(\mathbf{y}, \mathbf{z})_2 \in \mathcal{P}_M(\mathbf{x}^2)$, and $\lambda \in [0, 1]$ are given, then*

$$\lambda(\mathbf{y}, \mathbf{z})_1 + (1 - \lambda)(\mathbf{y}, \mathbf{z})_2 \in \mathcal{P}_M(\lambda \mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2).$$

[1] In this context, we accept the infinity value for derivatives.

By Lemma 1, it is easily shown that $\gamma(\alpha)$ is a non-increasing and concave function. Hence, if α is an interior point in $\mathcal{A}$ where $\gamma(\alpha)$ is finite, then the difference quotient in definition of γ' is a non-increasing function of h on $(0, +\infty)$ which is bounded from above, so $\gamma'(\alpha)$ exists and

$$\gamma'(\alpha) = \sup_{h>0} \frac{\gamma(\alpha + h) - \gamma(\alpha)}{h}\cdot$$

Moreover, $\gamma'_+(\alpha) \leq \gamma'_-(\alpha)$ where $\gamma'_+(\alpha)$ and $\gamma'_-(\alpha)$ are respectively right and left derivatives of γ.

For the purpose of this study, differentiability at $\alpha = 0$ is of interest. So, if 0 is an interior point of $\mathcal{A}$, then one-sided derivatives of γ exists at $\alpha = 0$ and the optimal values of model (14), or equivalently model (15), is finite. Otherwise, 0 is a left (right) endpoint of $\mathcal{A}$ so right (left) derivative $\gamma'_+(0)$ ($\gamma'_-(0)$) exists and the other derivative is ∞. However, to establish the derivatives, we need a fundamental theorem of optimal value, namely *envelopment theorem*.

Let $Q \subseteq R^n$ and $f : Q \times R \mapsto R$ be a differentiable function over R^{n+1}. Envelopment theorem simply states that the change in the optimal value of function $\varphi(a) = \sup_{\mathbf{q} \in Q} f(\mathbf{q}, a)$ with respect to the parameter $a \in R$ can be found by partially differentiating the objective function while holding $\mathbf{q}$ at its optimal value. That is, if $Q^*_a = \{\mathbf{q} \in Q : \varphi(a) = f(\mathbf{q}, a)\}$, then $\varphi'(a) = f_a(\mathbf{q}^*, a)$ for some $\mathbf{q}^* \in Q^*_a$ (see [31]).

Now, consider the Lagrangian optimal function on the part of environmental performance of model (3) at $P\big(\mathbf{x}_o^{+^*} + \alpha\mathbf{d}^i, \mathbf{y}_o^{+^*}, \mathbf{z}_o^{+^*} + \gamma(\alpha)\mathbf{d}^b\big)$ for any $\alpha \in \mathcal{A}$; *viz.*,

$$\begin{aligned}\mathcal{L}^*_e(\boldsymbol{\rho}^*, \alpha) = &\sum_{n=1}^{N} v_n^{2^*} \left(-\sum_{j=1}^{J} x_{nj}\eta_j + \left[x_{no}^{+^*} + \alpha d_n^i\right] \right) + \sum_{m=1}^{M} u_m^{2^*} \left(-\sum_{j=1}^{J} y_{mj}\eta_j + y_{mo}^{+^*} \right) \\ &+ \sum_{k=1}^{K} w_k^{2^*} \left(\sum_{j=1}^{J} z_{kj}\eta_j - \left[z_{ko}^* + \gamma(\alpha) d_k^b\right] \right) + u_0^{2^*} \left(\sum_{j=1}^{J} \eta_j - 1 \right)\end{aligned} \tag{17}$$

where $\boldsymbol{\rho}^* = (\mathbf{v}^{2^*}, \mathbf{u}^{2^*}, \mathbf{w}^{2^*}, u_0^{2^*})^t$ is the vector of optimal dual multipliers. Differentiating $\mathcal{L}^*_e(\boldsymbol{\rho}^*, \alpha)$ at $\alpha = 0$ by envelopment theorem and using the first order optimality condition yields:

$$\sum_{n=1}^{N} v_n^{2^*} d_n^i - \gamma'(0) \sum_{k=1}^{K} w_k^{2^*} d_k^b = 0$$

by which equation (13) for directional scale damage is immediately followed.

Finally, from differential calculus we know that for any $\alpha \in \mathcal{A}$ and $\alpha \neq 0$, $\gamma(\alpha) \approx \gamma'(0)\alpha$ provided that α is sufficiently close to 0. So, if we disturb inputs along direction $\mathbf{d}^i$ by a factor α, then the maximal response of bad outputs, for the target point remains still on the frontier, along direction $\mathbf{d}^b$ is given by $\mathrm{SD}(\mathbf{d})\alpha$.

Assuming $\mathrm{SD}(\mathbf{d}) > 1$. If the inputs are perturbed along $\mathbf{d}^i$ by α, the bad outputs should be perturbed along $\mathbf{d}^b$ by $\gamma(\alpha) \approx \mathrm{SD}(\mathbf{d})\alpha > \alpha$ for the target point still remains on the frontier. This intuitively proposes a qualitative view of scale damage, namely *damages to scale* as returns to scale in conventional literature:

Definition 1. If $\mathrm{SD}(\mathbf{d})$ is given in direction $\mathbf{d} = \big(\mathbf{d}^i, \mathbf{d}^b\big)$ on the frontier of $\mathcal{P}_M(\mathbf{x})$, then:

(a) $\mathcal{P}_M(\mathbf{x})$ prevails with *increasing damages to scale* (IDS) if $\mathrm{SD}^{\min}(\mathbf{d}) > 1$;
(b) $\mathcal{P}_M(\mathbf{x})$ prevails with *constant damages to scale* (CDS) if $\mathrm{SD}^{\min}(\mathbf{d}) \leq 1 \leq \mathrm{SD}^{\max}(\mathbf{d})$;
(c) $\mathcal{P}_M(\mathbf{x})$ prevails with *decreasing damages to scale* (DDS) if $\mathrm{SD}^{\max}(\mathbf{d}) < 1$.

It is possible to define other scale damage measures, such as non-increasing damage to scale, as a combination of constant and decreasing damages to scale. Regardless of their specific type, these measures generally assess how undesirable outputs respond to variations in inputs. For instance, expanding production units is not advisable when the output set exhibits increasing damage to scale, as this would likely lead to a significant increase in undesirable outputs.

TABLE 1. Case of Japanese electric power companies [22].

Firms	Input 1 (x_1)	Input 2 (x_2)	Desirable output 1 (y_1)	Desirable output 2 (y_2)	Undesirable output (z)
	Total assets (100 billion JPY)	No. of employees (1000)	Total sales (100 GWh)	No. of customers (100 000)	CO_2 emission (100 000 Tons)
DMU1	15.60	5.70	318.40	39.40	167.80
DMU2	36.80	12.40	811.00	76.80	397.90
DMU3	129.90	37.90	2889.60	284.90	1265.00
DMU4	51.10	16.20	1297.30	104.60	646.70
DMU5	14.20	4.60	281.50	20.80	185.20
DMU6	62.40	22.10	1458.70	134.00	549.90
DMU7	26.10	9.90	612.20	51.90	430.70
DMU8	13.50	6.00	287.00	28.30	114.60
DMU9	38.30	12.50	858.80	84.00	341.00
Ranges	0.0017	0.0060	0.0001	0.0008	0.0002

TABLE 2. UEM evaluation of Japanese electric power companies.

Firms	UEM $\mathcal{E}$	Slacks						
		s_1^{i-}	s_2^{i-}	s_1^{i+}	s_2^{i+}	s_1^{g}	s_2^{g}	s^{b}
DMU1	0.9797	0.0000	0.0000	3.8763	2.2677	0.0000	0.0000	0.0000
DMU2	0.9506	1.9087	1.1030	8.5249	4.0781	0.0000	0.4419	0.0000
DMU3	*1.0000*	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
DMU4	0.9165	0.0000	0.0000	20.4372	8.0388	0.0000	0.0000	0.0000
DMU5	0.9635	0.0000	0.0000	7.2309	4.0112	0.0000	0.0000	0.0000
DMU6	*1.0000*	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
DMU7	0.9138	0.0000	0.0000	22.9095	7.7913	0.0000	0.0000	0.0000
DMU8	*1.0000*	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
DMU9	*1.0000*	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

5. ILLUSTRATIONS

Example 1 ([22]). Table 1 summarizes the data set of Japanese electric power companies which consist of two inputs (*i.e.*, the total amount of assets and the number of employees), two desirable outputs (*i.e.*, the total amount of sales and the number of customers) and an undesirable output (*i.e.*, the total amount of CO_2 emission). The Japanese electric power industry, consists of nine investor-owned electric power firms, all of which are vertically integrated from generation to retail supply of electricity. Table 1 describes the results on their performances and Table 1 indicates the quantities for their scale measures. Note that our results differ with theirs because we assume weak disposability in the construction of output set $\mathcal{P}_W(\mathbf{x})$ in model (1).

Table 2 presents the performance evaluations of the Japanese electric power firms using the UEM model (3), highlighting optimal slacks, and global and individual efficiencies. The UEM quantifies the overall efficiency of a DMU relative to the most efficient DMU in the sample, with a UEM value of 1, italicized to indicate the highest efficiency. The presence of slack variables suggests opportunities for a DMU to increase its outputs or reduce its inputs without compromising efficiency.

Table 3 provides insights into the scale measures for the Japanese electric power companies, derived from relations (8) and (10). Scale elasticity indicates how changes in a company's scale size affect its efficiency,

TABLE 3. Scale measures for Japanese electric power companies.

Firms	Scale Elasticities		Scale Damages	
	SE^{min}	SE^{max}	SD^{min}	SD^{max}
DMU1	0.8504	3.8265	1.0332	1.2838
DMU2	0.0000	1.6869	1.0140	1.1197
DMU3	0.0000	1.0314	0.0000	$+\infty$
DMU4	0.7959	1.1014	1.1719	1.6964
DMU5	1.3960	$+\infty$	1.0301	1.2571
DMU6	0.9688	0.9688	0.0000	1.8189
DMU7	0.9375	0.9375	1.0129	1.1106
DMU8	1.1854	$+\infty$	0.0000	1.4156
DMU9	0.9375	0.9375	0.0000	0.3464

while scale damages assess the impact of these scale changes on the production of the undesirable output, CO_2 emissions.

Based on the results, several managerial implications can be drawn:

- *Efficiency:* DMUs 3, 6, 8, and 9 are identified as fully efficient, with UEM values equal to 1, indicating that they are operating at optimal levels. In contrast, DMUs 1, 2, 4, 5, and 7 have UEM scores less than 1, suggesting potential areas for efficiency improvement. Managers of these less efficient DMUs can utilize the slack variables from the extended RAM model (3) to pinpoint areas where outputs can be increased or inputs reduced to enhance overall efficiency.
- *Scale size*: as indicated in Table 3, companies 5 and 8, which exhibit $SE^{min} > 1$, demonstrate that after addressing inefficiencies and aligning with operational efficiency boundaries in model (3), a proportional increase in their inputs (Total Assets and Number of Employees) results in a more significant increase in their desirable outputs (Total Sales and Number of Customers). For these companies, a development scenario focused on operational expansion is recommended. Conversely, for companies 6, 7, and 9, where $SE^{max} < 1$, an expansion focused on operational size is not advisable.
  For the remaining companies, where $SE^{min} < 1 < SE^{max}$, the response of desirable outputs to the proportional increases in inputs will vary. However, for companies 1 and 4, where SE^{min} is close to 1, an operational development scenario could be considered as a viable option.
- *Environmental impact*: a similar interpretation applies to SD values, with values less than 1 indicating superior environmental performance. Companies 1, 2, 4, 5, and 7, with $SD^{min} > 1$, are likely to experience worsening environmental outcomes with proportional increases in all inputs, primarily due to higher CO_2 emissions as the undesirable output. In contrast, for company 9, with $SD^{max} < 1$, the development scenario would result in the least adverse environmental impact per unit increase in both inputs.

To integrate these findings, managers can consider developing various strategic scenarios, such as cautious or optimistic approaches. In a cautious scenario, the lowest SE values and highest SD values could serve as criteria for prioritizing units for development. Managers should also focus on implementing measures to reduce CO_2 emissions to enhance efficiency and mitigate environmental impact. Companies with the highest SE-to-SD ratio under stringent conditions, such as companies 9 and 5, should be prioritized in the development plan. The radar chart in Figure 1 is particularly useful for visualizing these scenarios.

Example 2. Reconsider Example 1 as a matter of comparing conventional with directional scale damage. The results are given in Table 4 for directions $(\mathbf{x}_{1o}^{+^*}, \mathbf{x}_{2o}^{+^*}, \mathbf{z}_o^{+^*})^t$ and $(0.03, 0.01, 0.29)^t$. The latter direction is produced randomly.

Table 4 presents the results for directional scale damage (SD) for the Japanese electric power companies, derived using the linear optimization model (15) and relation (13). The table displays the minimum and maxi-

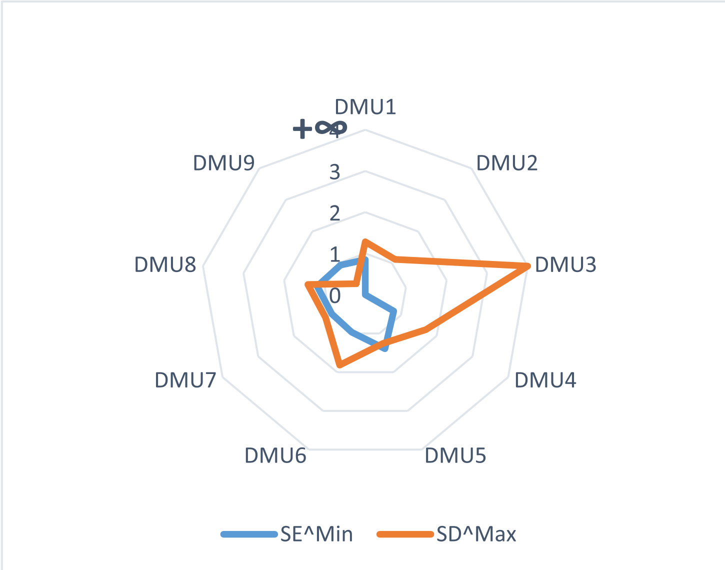

FIGURE 1. Operational and environmental trade-offs in cautious development scenarios.

TABLE 4. Directional scale damages for Japanese electric power companies.

Firms	$\mathbf{d} = (x_{1o}^{+^*}, x_{2o}^{+^*}, z_o^{+^*})^t$		$\mathbf{d} = (0.03, 0.01, 0.29)^t$	
	$\mathrm{SD}^{\min}(\mathbf{d})$	$\mathrm{SD}^{\max}(\mathbf{d})$	$\mathrm{SD}^{\min}(\mathbf{d})$	$\mathrm{SD}^{\max}(\mathbf{d})$
DMU1	1.0332	1.2838	0.9769	1.0009
DMU2	1.0140	1.1197	0.9769	1.0009
DMU3	0.0000	$+\infty$	0.0000	$+\infty$
DMU4	1.1719	1.6964	1.1626	1.6754
DMU5	1.0301	1.2571	0.9769	1.0009
DMU6	0.0000	1.8189	0.0000	1.6754
DMU7	1.0129	1.1106	0.9769	1.0009
DMU8	0.0000	1.4156	0.0000	1.0009
DMU9	0.0000	0.3464	0.0000	0.3385

mum SD values under two different perturbation directions: the standard direction $\mathbf{d} = (x_{1o}^{+^*}, x_{2o}^{+^*}, z_o^{+^*})^t$ and a randomly generated direction $\mathbf{d} = (0.03, 0.01, 0.29)^t$.

A comparison between the conventional SD measures from Table 3 and the directional SD measures from Table 4 highlights that the directional approach extends the conventional scale damage assessment. The results underscore that the choice of perturbation direction $\mathbf{d}$ significantly influences the evaluation of target points. For instance, in Firm 2, the direction change results in a shift from IDS to CDS.

The results also indicate that for some DMUs, the directional scale damage values diverge from the conventional ones. This divergence arises from the differing input mix changes induced by the selected directions in relations (12) and (13), as opposed to the proportional changes in the conventional measure.

For instance, Firm 3 consistently shows an $\mathrm{SD}^{\min}$ value of 0 across both perturbation directions, indicating that its range is not sensitive to these directional changes. In contrast, other firms exhibit slight variations; for example, Firm 1 shows a decrease in $\mathrm{SD}^{\min}$ from 1.0332 to 0.9769, a reduction of approximately 5%, though the overall change is relatively minor.

Based on the results in Table 4 under the standard direction $\mathbf{d} = (x_{1o}^{+^*}, x_{2o}^{+^*}, z_o^{+^*})^t$, the firms can be classified as follows:

– IDS: DMU1, DMU2, DMU4, DMU5, DMU7

These firms exhibit increasing damages to scale, indicated by minimum SD values greater than 1 in the given direction.

– CDS: DMU3

This firm demonstrates constant damages to scale, where the minimum SD value is less than or equal to 1, and the maximum SD value is greater than or equal to 1 in the same direction.

– DDS: DMU6, DMU8, DMU9

These firms show decreasing damages to scale, as their maximum SD values are less than 1 in the given direction.

The managerial interpretation of these results emphasizes the flexibility of the new approach in selecting perturbation directions for real-world applications. By evaluating the elasticity of DMUs in chosen directions, the directional scale damage measure allows for a more tailored analysis. This means that when applying directional scale damage measures, careful selection of the perturbation direction is crucial to ensure that the results accurately reflect practical changes. Additionally, this approach provides a more comprehensive assessment of DMU performance by evaluating how input changes influence undesirable outputs in various directions.

5.1. Emprical application

In agricultural research, DEA elasticity measures have been pivotal in enhancing productivity and efficiency. Qi *et al.* [40] explored the relationship between farmland costs and the scale efficiency of agricultural production, with a particular emphasis on the impact of farmland price deviations. Through a threshold regression model, they analyzed how these costs affect agricultural efficiency, offering valuable insights for optimizing competitiveness by addressing price disparities in farmland. Similarly, Anang [2] conducted an in-depth analysis of the technical and scale effects in groundnut production in northern Ghana using DEA modeling. This study evaluated efficiency scores under various assumptions, identifying key factors that influence both technical and scale elasticity, and providing crucial guidance for policymakers seeking to improve efficiency in groundnut production.

Our research applies DEA scale elasticity measures to the horticultural industry in Qazvin province, Iran – a sector that plays a significant role in the nation's economy. Our approach is innovative in that it considers both desirable and undesirable outputs, including environmental concerns such as pollutants and waste. This comprehensive assessment offers a holistic evaluation of the horticultural industry's performance in Qazvin, providing actionable insights to enhance both efficiency and sustainability.

5.1.1. Data and results

To demonstrate the practical application of our proposed models, we apply the approach to a dataset consisting of 35 farmers in Qazvin province, Iran. This application aims to enhance our understanding of the production processes within the horticultural sector, providing valuable insights into both its performance and its environmental impact, particularly in relation to directional damage to scale.

To boost production and create employment opportunities for the region's unemployed youth, the Qazvin Agricultural and Natural Resources Research and Education Center – a branch of the Horticultural Sciences Research Institute in Iran – has launched a transformative initiative. This project involves the development of approximately 350 hectares of land, which has been allocated to 35 farmers for the cultivation of cherry, sour cherry, and walnut trees, as illustrated in Figure 2.

Beyond enhancing agricultural output, a key objective of this venture is to mitigate environmental impacts by closely monitoring scale damage associated with increased production. By establishing conversion industries,

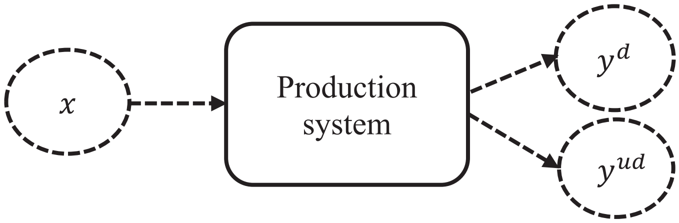

FIGURE 2. Horticulture production unit.

the project aims to add value to the produced goods, reduce wastage, and address potential environmental repercussions such as pollution and resource depletion.

Within the scope of this ambitious project, the evaluation indicators are systematically categorized to assess both the economic and environmental impacts, with particular attention to scale damage. To provide a comprehensive overview of the multifaceted evaluation criteria essential for achieving the project's goals, including sustainable agricultural practices and environmental preservation, we consider the following input and output factors:

Inputs ($\boldsymbol{x}$)

- Water consumption (liter): x_1
- Fertilizer consumption (kg): x_2
- Manpower (person): x_3
- Seedlings (kg): x_4

Desirable outputs ($\boldsymbol{y}$)

- Cherry compote (ton): y_1
- Sour cherry compote (ton): y_2
- Packaged walnut kernels (ton): y_3

Undesirable outputs ($\boldsymbol{z}$)

- Cherry seeds (ton) z_1
- Sour cherry seeds (ton): z_2
- Walnut shell (ton): z_3

Table 5 presents comprehensive data on inputs, desirable outputs, and undesirable outputs for 35 farmers. In the initial step, we evaluate the efficiency of these farmers using the extended RAM model (3). To estimate the degree of scale damage, we account for the impact of directional changes in inputs on the production of undesirable outputs. It is essential to consider the environmental implications of the project to minimize these undesirable outputs during the assessment.

Using the evaluation model (15) outlined in the previous section, we compute the scale damage measure for each farmer, considering their resource utilization and ability to convert inputs into undesirable outputs. The DTS results are provided in Table 4 for the standard direction $(\mathbf{x}^*_{1o}, \mathbf{x}^*_{2o}, \mathbf{x}^*_{3o}, \mathbf{x}^*_{4o}, \mathbf{x}^*_{5o}, \mathbf{z}^*_{1o}, \mathbf{z}^*_{2o}, \mathbf{z}^*_{3o})^t$.

Table 6 highlights both the minimum and maximum values of the Scale Damage for the standard perturbation direction. The last column of this table presents the directional DTS status of the production units. This analytical approach is crucial for identifying the impact of production size, particularly when environmental factors are considered.

5.1.2. Discussions and managerial interpretations

The computed directional scale damage values enable a detailed analysis of the farms, facilitating for a comprehensive interpretation of scale damage across the entire dataset. The results are illustrated in Figure 3.

TABLE 5. Inputs/Outputs data for 35 farmers in Qazvin province.

DMU	Inputs				Desirable products			Undesirable products		
	x_1	x_2	x_3	x_4	y_1	y_2	y_3	z_1	z_2	z_3
1	4 000 000	900	200	4500	3	3.6	0.075	0.45	0.45	0.225
2	4 000 000	1000	200	4500	8	10	0.2	1.2	1.25	0.6
3	4 000 000	1000	2000	4500	24	28	0.375	3.6	3.5	1.13
4	4 000 000	1000	2000	4500	28	32	0.75	4.2	4	2.3
5	4 000 000	1000	2000	4500	32	36	1.25	4.8	4.5	3.75
6	4 000 000	900	2010	4600	3.25	3.5	0.05	0.36	0.38	0.3
7	4 000 000	900	2030	4600	8	9.5	0.15	1.3	1.2	0.5
8	4 000 000	950	2040	4600	23.5	27.5	0.375	3.6	3.25	1.2
9	3 980 000	970	2050	4600	28.5	32.5	0.8	3.9	3.75	3
10	3 980 000	1000	2070	4600	31.5	35	1.2	5.1	4.8	4.5
11	4 500 000	910	2100	4700	3.2	3.45	0.05	0.33	0.4	0.34
12	4 500 000	920	2100	4700	7.5	10	0.155	1.2	1.25	0.6
13	4 500 000	900	2100	4700	24	26.5	0.39	3.3	3.4	1.13
14	4 500 000	900	2100	4700	28	32	0.805	3.6	3.8	3.08
15	4 500 000	900	2100	4700	31	35.5	1.215	4.8	4.9	4.05
16	4 489 000	905	2114	4600	3.15	3.4	0.045	0.3	0.4	0.4
17	4 500 000	915	2200	4700	7.55	9.5	0.15	1.14	1.3	0.7
18	4 600 000	926	2200	4700	24.45	25.5	0.4	3.06	3.3	0.98
19	4 600 000	930	2200	4750	27.8	32.5	0.8	3.5	3.5	3
20	4 600 000	910	2160	4700	30.5	35	1.25	4.6	5	4.13
21	4 529 000	890	2200	4610	3	3.35	0.05	0.27	0.43	0.45
22	4 600 000	900	2250	4600	7.5	9	0.14	1.05	1.3	0.75
23	4 610 000	910	2250	4600	24.5	25	0.3625	3	3.5	1.05
24	4 650 000	915	2270	4600	26.5	31	0.755	3.3	3.8	2.9
25	4 500 000	915	2280	4600	30	36.5	1.1955	4.5	4.8	3.8
26	4 200 000	859	2150	4600	2.5	3.25	0.045	0.24	0.48	0.5
27	4 100 000	910	2169	4720	7	8.5	0.15	0.96	1.25	0.9
28	4 100 000	920	2170	4700	25	22.5	0.3745	2.8	3.4	1.2
29	4 000 000	900	2180	4650	27	30	0.76	3.2	3.7	3
30	4 000 000	900	2100	4600	29.5	35	1.2005	1.05	4.8	3.9
31	4 000 000	812	2100	4520	2.75	3	0.05	0.3	0.5	0.08
32	4 200 000	920	2170	4500	6.5	9	0.155	1.05	1.5	0.75
33	4 100 000	930	2185	4500	24.5	21	0.375	3	3.5	1.2
34	4 100 000	935	2200	4500	29	29.5	0.81	3.3	3.8	3.8
35	4 000 000	935	2125	4500	31	37.5	1.285	4.2	5	4.5

In our analysis of the 35 farmers in Qazvin province, we closely examined specific units to understand the economic implications of their DTS status. As detailed in Table 6, the results provide valuable insights into production efficiency and how these farmers respond to changes in scale. The SD values range from 0 to over 1, reflecting varying patterns of scale damage among the farmers.

Most farmers – 29 out of 35 – exhibit decreasing scale damage ($\mathrm{SD}^{\max} < 1$), suggesting that technological innovation is not is not immediately necessary, and a slight expansion in the size of operations could be beneficial. For instance, DMUs 3 and 4, with SD values near zero, demonstrate resilience to changes in inputs related to undesirable outputs, highlighting potential for further operational improvements. On the other hand, some farms exhibit constant scale damage ($\mathrm{SD}^{\min} \leq 1 \leq \mathrm{SD}^{\max}$), indicating that they are operating at an optimal scale, where output remains balanced despite changes in input direction.

TABLE 6. Directional scale damages for 35 farmers in Qazvin province.

Farmers	$\mathbf{d} = (\mathbf{x}_{1o}^*, \mathbf{x}_{2o}^*, \mathbf{x}_{3o}^*, \mathbf{x}_{4o}^*, \mathbf{x}_{5o}^*, \mathbf{z}_{1o}^*, \mathbf{z}_{2o}^*, \mathbf{z}_{3o}^*)^t$		DTS status
	$\mathrm{SD}^{\min}(\mathbf{d})$	$\mathrm{SD}^{\max}(\mathbf{d})$	
DMU1	0.7083	0.9937	DDS
DMU2	0.3470	0.5800	DDS
DMU3	0	0.2728	DDS
DMU4	0	0.2170	DDS
DMU5	0	0.18570	DDS
DMU6	0.8116	1	CDS
DMU7	0.3787	0.5614	DDS
DMU8	0	0.2660	DDS
DMU9	0	0.2067	DDS
DMU10	0	0.1818	DDS
DMU11	0.8468	1.1023	CDS
DMU12	0.4000	0.5650	DDS
DMU13	0	0.2540	DDS
DMU14	0	0.1985	DDS
DMU15	0	0.1682	DDS
DMU16	0.9298	1	CDS
DMU17	0.4070	0.5456	DDS
DMU18	0	0.2640	DDS
DMU19	0	0.2136	DDS
DMU20	0	0.1716	DDS
DMU21	0.5889	1	CDS
DMU22	0.4214	0.5526	DDS
DMU23	0	0.2654	DDS
DMU24	0	0.2159	DDS
DMU25	0	0.1819	DDS
DMU26	0.6021	1	CDS
DMU27	0	0.5508	DDS
DMU28	0	0.2602	DDS
DMU29	0	0.2085	DDS
DMU30	0	0.3070	DDS
DMU31	1	1.0033	CDS
DMU32	0	0.5417	DDS
DMU33	0	0.2667	DDS
DMU34	0	0.2048	DDS
DMU35	0	2.1417	CDS

Importantly, none of the units fall into the category of increasing scale damage, suggesting that no farms need to reduce their operational size under current technology. This finding highlights the overall environmental efficiency of the horticultural industry in Qazvin province. However, DMU 35 stands out with a significantly high SD value of 2.14, indicating a heightened risk of producing more undesirable outputs with input changes in the natural direction. For this unit, it is advisable for managers to carefully monitor any marginal increases in operational size and consider technological innovations to prevent excessive production of undesirable outputs, thereby maintaining environmental impact within CDS ranges and ensuring sustainability.

Policy Implications: we recommend disseminating best practices from farms with lower SD values to those with higher SD values to enhance overall performance. Additionally, farms with higher SD values should implement targeted environmental management practices to mitigate risks associated with increased undesirable

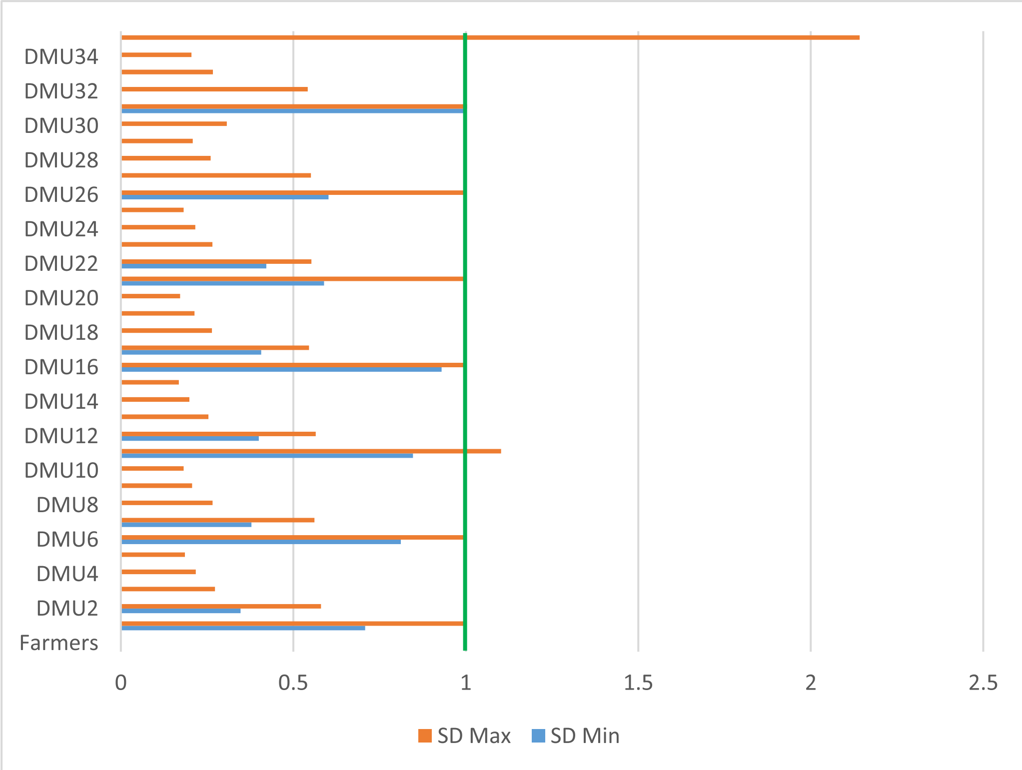

FIGURE 3. Scale damage values for the 35 farmers.

outputs, with technological innovation as a potential strategy. Moreover, exploring opportunities for resource optimization and input adjustments is essential to minimizing scale damage.

These findings empower stakeholders and policymakers to make informed managerial decisions aimed at enhancing overall performance and sustainability in the horticultural industry. Adjustments and technological improvements can be tailored to the specific characteristics of each farm, ensuring a more effective approach.

6. Conclusion

In conclusion, this study introduces an extended method for efficiency measurement that captures the directional impact of inputs on both desirable and undesirable outputs. By adapting the RAM, we provide a unified efficiency score that evaluates environmental and performance aspects concurrently. Our extension of scale elasticity and scale damage to directional scale measures allows for a detailed examination of how target points on the efficiency frontier are influenced by partial movements.

We have thoroughly explored the theoretical framework of this extension, offering linear programming formulations and numerical examples to support its computation. Our models deliver a comprehensive assessment of firm performance, integrating both environmental and operational dimensions. The directional scale measures provide nuanced insights into managing undesirable outputs, enhancing our understanding of efficiency in various contexts.

The practical application of our approach to 35 horticulture farms in Qazvin Province, Iran, confirms its real-world effectiveness. By integrating environmental and performance dimensions into a unified efficiency score, our method accurately assessed farm productivity, pinpointed areas for improvement, and facilitated targeted development strategies based on the DTS classifications. This underscores the importance of combining operational and environmental factors for informed decision-making in agriculture.

Future research could investigate alternative efficiency measurement models, particularly those incorporating non-discretionary variables like farming land area. Additionally, expanding our approach to estimate directional damage to scale using SBM [50] and BAM [10] models could further elucidate the methodology's broader applicability and nuances, enhancing our understanding of efficiency and its implications for decision-making and resource allocation.

In summary, this study advances the field of efficiency measurement by integrating operational and environmental assessments through unified calculations of directional scale elasticity and scale damage. Continued research in this area promises to deepen our understanding of efficiency measurement and its practical applications across diverse sectors.

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