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THE DIFFERENTIAL ON GRAPH OPERATOR R(G)

Ludwin Ali Hernández Basilio[1], Jesús Leaños Macías[2], Omar Rosario Cayetano[1,*] and José María Sigarreta Almira[1]

Abstract. Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let S be a subset of $V(G)$, and let $B(S)$ be the set of neighbours of S in $V(G) \setminus S$. The differential $\partial(S)$ of S is the number $|B(S)| - |S|$. The maximum value of $\partial(S)$ taken over all subsets $S \subseteq V(G)$ is the differential $\partial(G)$ of G. The graph $\mathrm{R}(G)$ is defined as the graph obtained from G by adding a new vertex v_e for each $e \in E(G)$, and by joining v_e to the end vertices of e. In this paper we study the relationship between $\partial(G)$ and $\partial(\mathrm{R}(G))$, and give tight asymptotic bounds for $\partial(\mathrm{R}(G))$. We also exhibit some relationships between certain vertex sets of G and $\mathrm{R}(G)$ which involve well known graph theoretical parameters.

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1. Introduction

Social networks, such as Facebook or Twitter, have served as a powerful tool for communication and information disseminating. As a consequence of their massive popularity, social networks now have a wide variety of applications in the viral marketing of products and in political campaigns. Motivated by their numerous applications, some authors [32,33] have proposed several influential maximization problems, which share a fundamental algorithmic problem for information diffusion in social networks: the problem of determining the best group of nodes to influence the rest. As it was showed in [6], the study of the differential $\partial(G)$ of a graph G could be motivated by such scenarios. Before moving on any further, let us introduce the basic definitions and notation that will be used in this paper.

Throughout this paper, $G = (V(G), E(G))$ is a simple connected graph of order $n \geq 2$ with vertex set $V(G)$ and edge set $E(G)$. For the rest of this section, let us assume that $v \in V(G)$ and that $S \subseteq V(G)$. The set of all vertices of G that are adjacent to $v \in V(G)$ is the *neighborhood of* v and is denoted by $N_G(v)$. The set $N_G(v) \cup \{v\}$ is the *closed neighborhood of* v and is denoted by $N_G[v]$. We define $N_G(S) := \bigcup_{v\in S} N_G(v)$ and $N_G[S] := N_G(S) \cup S$. We shall use $\overline{S}$ to denote $V(G) \setminus S$, and $N_{\overline{S}}(v)$ to denote the set of all vertices of $\overline{S}$ that are adjacent to v. An *external private neighbour of* $v \in S$ *with respect to* S is a vertex $w \in N_{\overline{S}}(v)$ such that $w \notin N_G(u)$ for any $u \in S \setminus \{v\}$. The set of all external private neighbors of v with respect to S is denoted by

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[1] Facultad de Matemáticas, Universidad Autónoma de Guerrero, Carlos E. Adame No.54 Col. Garita, 39650 Acalpulco Gro., Mexico.

[2] Academic Unit of Mathematics, Autonomous University of Zacatecas. Paseo la Bufa, int. Calzada Solidaridad, 98060 Zacatecas, Mexico.

*Corresponding author: orosario@uagro.mx

$\text{epn}_G[v, S]$. The *degree* $\delta_G(v)$ of v is the number $|N_G(v)|$. As usual, the *minimum* and *maximum degree* of G will be denoted by $\delta(G)$ and $\Delta(G)$, respectively. The subgraph of G induced by S will be denoted by $\langle S\rangle_G$.

From now on, we shall use $B_G(S)$ to denote $N_G(S)\setminus S$, and $C_G(S)$ to denote $V(G)\setminus(B_G(S)\cup S)$. Then $V(G)$ is disjoint union of $S, B_G(S)$ and $C_G(S)$. The *differential* $\partial_G(S)$ of $S \subseteq V(G)$ is the number $|B_G(S)| - |S|$, and the *differential* $\partial(G)$ of G is the maximum value of $\{\partial_G(S) : S \subseteq V(G)\}$. We call $S \subseteq V(G)$ a ∂-*set* or a *differential set* if $\partial_G(S) = \partial(G)$. If S is a differential set, and it has minimum (maximum) cardinality among all differential sets of G, then we call it a *minimum (maximum) differential set*. From the definition of $\partial(G)$ it is easy to see that if G is disconnected, and $G_1, \ldots, G_k$ are its connected components, then $\partial(G) = \partial(G_1) + \cdots + \partial(G_k)$. In view of this, all graphs under consideration in this paper will be connected unless otherwise stated.

As long as no confusion can arise, we will omit the reference to G in $N_G(v)$, $N_G[v]$, $N_G(S)$, $N_G[S]$, $\text{epn}_G[v, S]$, $\delta_G(v)$, $\langle S\rangle_G, B_G(S), C_G(S), \partial_G(S)$, etc.

In [13, 14, 16, 17, 35] have been studied $\partial(G)$, together with a variety of other kinds of differentials of a set. Since then, the differential of a graph has received considerable attention [1–4, 6, 8–10, 19, 42, 47]. The differential of a set $S \subseteq V(G)$ was previously considered in [27], where it was denoted by $\eta(S)$. In 2012, Bermudo and Fernau [8] showed a close relationship between $\partial(G)$ and the Roman domination number $\gamma_R(G)$ of G, namely, $\partial(G) + \gamma_R(G) = n$. On the other hand, the minimum differential of an independent set was considered in [51], and the B-differential (or enclaveless number) of G, defined as $\psi(G) := \max\{|B(S)| : S \subseteq V(G)\}$, was investigated in [35, 49].

A *graph operator* is a mapping $F : \Lambda \to \Gamma$, where Λ and Γ are families of graphs. The study of how a graph invariant changes when a graph operator is applied is a classical problem in graph theory. In [34], Krausz introduced the concept of graph operators and defined four graph operators denoted by $\mathrm{Q}(G), \mathrm{R}(G), \mathrm{S}(G)$, and $\mathrm{T}(G)$. As can be seen in the literature, these operators have received considerable attention from both theoretical and practical perspectives. Thus, different types of these operators have been investigated in studies related to dynamic networks (see [28, 40, 41]), mathematical chemistry (see [12, 24, 25, 43–45]), discrete geometry (see [20, 36, 37]), and domination theory (see [11, 15, 18, 23, 38, 39, 46, 52]).

The study of parameters of graph operators has attracted the attention of several authors who seek to better understand how the properties of a graph and its operators can affect certain parameters of the graph. Furthermore, since the determination of the differential of a graph is a problem NP-complete (see [7]), it is crucial to obtain bounds for the differential of these operators on graphs.

The main goal of our project is to answer the following question: what information can we obtain about $\partial(\mathrm{R}(G))$ if we know certain properties on either G or $\partial(G)$?

In this work we deal only with the graph operator $\mathrm{R}(G)$. Following [22], $\mathrm{R}(G)$ is defined as the graph obtained from G by adding a new vertex v_e for each edge e of G, and by joining v_e to the end vertices of e. See Figure 1 for an example. We begin our project with the study of $\mathrm{R}(G)$ because it shares structural properties with each of the other three operators. In view of this, we naturally hope that the present work gives useful knowledge for the study of the rest of the operators.

The rest of this paper is organized as follows. In Section 2 we exhibit several interesting connections between G and $\mathrm{R}(G)$, which involve vertex sets that are either differential, dominating or vertex cover, for at least one of G or $\mathrm{R}(G)$. Additionally, we determine the exact value of $\partial(\mathrm{R}(G))$ for some infinite families of graphs. Finally, in Section 3 we establish tight lower and upper bounds for $\partial(\mathrm{R}(G))$ in terms of the independence number of G and the maximum cardinality of a differential set of $\mathrm{R}(G)$.

2. The differential of the graph $\mathrm{R}(G)$

In this section we report some results on $\partial(\mathrm{R}(G))$. Most of such results show relations between certain vertex sets of G and $\mathrm{R}(G)$. Each vertex set involved in these results is either a differential set, a dominating set, or a vertex cover set in at least one of G or $\mathrm{R}(G)$.

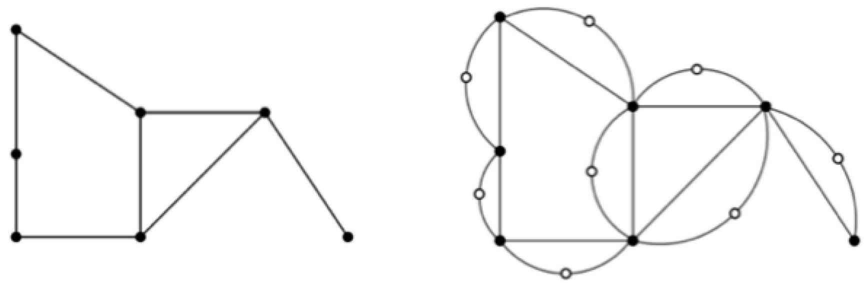

FIGURE 1. The graph on the *right* is the corresponding $\mathrm{R}(G)$ of the graph G on the *left*.

We start by listing some basic properties of $\mathrm{R}(G)$, which can be deduced easily from the definition of $\mathrm{R}(G)$. We use E_n to denote the graph of order $n \geq 2$ with no edges.

Proposition 2.1. *Let $G = (V(G), E(G))$ be a graph of order n, and let $\mathrm{R}(G)$ be as above. Then*

(i) $|V(\mathrm{R}(G))| = |V(G)| + |E(G)|$.
(ii) $|E(\mathrm{R}(G))| = 3|E(G)|$.
(iii) *G is an induced subgraph of $\mathrm{R}(G)$.*
(iv) *$G \cong \mathrm{R}(G)$ if and only if $G \cong E_n$.*
(v) *If $v \in V(G)$, then $\delta_{\mathrm{R}(G)}(v) = 2\delta_G(v)$.*
(vi) *G is connected and simple if and only if $\mathrm{R}(G)$ is connected and simple.*

In view of Proposition 2.1(iii), we can partition $V(\mathrm{R}(G))$ into two subsets: the set of vertices V formed by the vertices of $\mathrm{R}(G)$ that are in G, and $U := V(\mathrm{R}(G)) \setminus V$. Thus the subgraph $\langle V \rangle$ of $\mathrm{R}(G)$ is isomorphic to G, $|U| = |E(G)|$, and any vertex u in U has exactly two neighbours, say v_1 and v_2, such that $v_1, v_2 \in V$ and $v_1v_2 \in E(G)$. Such a partition $\{V, U\}$ will be called the *canonical partition* of $V(\mathrm{R}(G))$.

We recall that a *vertex cover* of G is a subset $S \subseteq V(G)$ such that every edge of G has at least one end vertex in S. The *vertex cover number* of G is the size of any smallest vertex cover in G and is denoted by $\tau(G)$. A subset $S \subseteq V(G)$ is a *dominating set* of G if each vertex in $V(G)$ is in S or is adjacent to a vertex in S. The *domination number* of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$. For more information on domination in graphs see [29,30]. Similarly, a subset $S \subseteq V(G)$ is an *independent set* of G if any two distinct vertices in S are not adjacent in G. The *independence number* of G is the maximum cardinality of an independent set of G and is denoted by $\alpha(G)$.

Proposition 2.2. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. Then V contains a minimum dominating set of $\mathrm{R}(G)$.*

Proof. Let W be a dominating set of $\mathrm{R}(G)$ such that $\gamma(\mathrm{R}(G)) = |W|$. Since if $W \subseteq V$ then there is nothing to prove, we assume that $W \cap U$ has at least one vertex u. We know that u has exactly two neighbours v_1 and v_2 such that $v_1, v_2 \in V$ and $v_1v_2 \in E(G)$. We can assume that $\{v_1, v_2\} \cap W = \emptyset$, as otherwise $W \setminus \{u\}$ is a dominating set of $\mathrm{R}(G)$, contradicting the minimality of W. Then $W' := (W \setminus \{u\}) \cup \{v_1\}$ is a minimum dominating set of $\mathrm{R}(G)$ with $|W' \cap U| = |W \cap U| - 1$. Clearly, continuing in this way, we can construct the required dominating set of $\mathrm{R}(G)$. $\square$

Proposition 2.3. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. If D is a differential set of $\mathrm{R}(G)$, then V contains a differential set S of $\mathrm{R}(G)$ such that $|S| = |D|$.*

Proof. Let D be a subset of $V(\mathrm{R}(G))$ such that $\partial(\mathrm{R}(G)) = \partial(D)$. Since if $D \subseteq V$ then there is nothing to prove, we assume that $D \cap U$ has at least one vertex u. Again, we know that u has exactly two neighbours v_1 and v_2

such that $v_1, v_2 \in V$ and $v_1v_2 \in E(G)$. Then $N[u] \subseteq N[v_i]$ for any $i \in \{1,2\}$. We note that if $\{v_1, v_2\} \cap D \neq \emptyset$, then $\partial(D \setminus \{u\}) \geq \partial(D) + 1 = \partial(\mathrm{R}(G)) + 1$, which is impossible. Then we must have that $\{v_1, v_2\} \cap D = \emptyset$ Let $D' = (D \setminus \{u\}) \cup \{v_1\}$. Then $|D'| = |D|$, $|B(D)| \leq |B(D')|$, and so $\partial(D') = \partial(\mathrm{R}(G))$. Also note that $|D' \cap U| = |D \cap U| - 1$. Continuing in this way, we can construct the required S. □

Proposition 2.4. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. If $S \subseteq V$ is a differential set of $\mathrm{R}(G)$, then S can be extended to a differential set S' of $\mathrm{R}(G)$ such that: (i) $S' \subseteq V$, and (ii) S' is a dominating set of G.*

Proof. Let S be as in the hypothesis of Proposition 2.4. We recall that $V(\mathrm{R}(G))$ is disjoint union of $S, B(S)$ and $C(S)$. If S is a dominating set of G, then take $S' = S$ and we are done. Then we can assume that there exists a vertex $v_1 \in C(S) \cap V$. Since G is a connected graph of order at least 2, then v_1 has at least one neighbour $v_2 \in V$. If $v_2 \in C(S)$, then the unique vertex $u \in U$ such that $N(u) = \{v_1, v_2\}$ must belong to $C(S)$. But then we have, $\partial(S \cup \{v_1\}) > \partial(S) = \partial(\mathrm{R}(G))$, a contradiction. Then we can assume that all the neighbours of v_1 in V are in $B(S)$. In particular, we have that $v_2 \in B(S)$, and then $u \in C(S)$. Note that $\partial(S \cup \{v_1\}) = \partial(S) = \partial(\mathrm{R}(G))$. Continuing in this way, we can construct the required S'. □

Corollary 2.5. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. Then V contains a differential set S of $\mathrm{R}(G)$ such that S is a dominating set of G.*

We remark that the set S guaranteed by Corollary 2.5 is not necessarily a minimum dominating set. Suppose that S is the smaller part of the vertex bipartition of $V(K_{p,q})$. As we shall see later, if $|S| = p < q$, then such an S is the only differential set of $\mathrm{R}(K_{p,q})$. According to Corollary 2.5, S is a dominating set for $K_{p,q}$, but it is far from being a minimum dominating set of $K_{p,q}$ (see Fig. 3).

Proposition 2.6. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. If $S \subseteq V$ is a differential set of $\mathrm{R}(G)$ and $\delta(G) \geq 2$, then S is a dominating set of G.*

Proof. Let S be as in the hypothesis of Proposition 2.6. Since if S is a dominating set of G then there is nothing to prove, we assume that V contains a vertex w that is not adjacent to any vertex of S. Let v_1 and v_2 be two distinct neighbours of w in G. Then $\mathrm{R}(G)$ has two distinct vertices $u_1, u_2 \in U$ such that $N(u_1) = \{w, v_1\}$ and $N(u_2) = \{w, v_2\}$. If $u_i \in N(S)$ for some $i \in \{1,2\}$, then we must have $v_i \in S$. This implies $w \in N(S)$, contradicting the choice of w. Thus we can assume that $u_i \in C(S)$ for $i = 1, 2$, and so $\partial(S \cup \{w\}) > \partial(S) = \partial(\mathrm{R}(G))$. Since this inequality is false, we conclude that such a w does not exist, as required. □

We remark that the reverse implication in Proposition 2.6 does not hold. Again, consider two vertices x and y in $K_{p,q}$ which belong to distinct parts of the vertex bipartition of $V(K_{p,q})$. Clearly, if $3 \leq p \leq q$, then $K_{p,q}$ has minimum degree at least 2 and $S = \{x, y\}$ is a (minimum) dominating set of $K_{p,q}$. However, S is not a differential set of $\mathrm{R}(K_{p,q})$. On the other hand, the set S of gray vertices of the graph in Figure 2 shows that the condition $\delta(G) \geq 2$ in Proposition 2.6 is necessary.

Corollary 2.7. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. Let $X, Y \subseteq V$ be differential sets of G and $\mathrm{R}(G)$, respectively. If $\delta(G) \geq 2$, then $|Y| \geq |X|$.*

Proof. Let X and Y be as in the hypothesis of Corollary 2.7. By Proposition 2.6, Y is a dominating set of G, and hence $\partial_G(X) \geq \partial_G(Y) = n - 2|Y|$. Since $\partial_G(X) \leq n - 2|X|$, then $|X| \leq |Y|$. □

The following observation can be deduced easily from the definition of $\partial(G)$ and is a key idea behind the proofs of Propositions 2.9 and 2.10.

Observation 2.8. If $\partial(G) \geq n - k$ for some positive integer k, and S is a differential set of G, then $|S| \leq \lfloor k/2 \rfloor$.

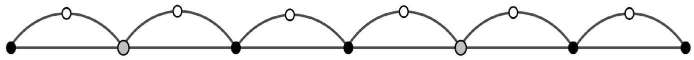

FIGURE 2. The minimum degree of P_7 is one, and the set S of gray vertices form a differential set for both P_7 and $\mathrm{R}(P_7)$. However, S is not a dominating set of P_7.

Proposition 2.9 ([10]). *Let G be a graph of order n and maximum degree Δ, then*

(a) $\Delta = n-1$ *if only if* $\partial(G) = n-2$.
(b) $\Delta = n-2$ *if only if* $\partial(G) = n-3$.
(c) *If* $\Delta = n-3$, *then* $\partial(G) = n-4$.

Proposition 2.10. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. If $n \geq 3$ and $m = |V(\mathrm{R}(G))|$, then the following statements hold.*

(i) $\partial(\mathrm{R}(G)) = m-2$ *if and only if* $G = K_{1,n-1}$.
(ii) $\partial(\mathrm{R}(G)) = m-3$ *if and only if* $G = K_{1,n-1} \cup \{e\}$, *where e is an edge joining two leaf vertices of* $K_{1,n-1}$.

Proof. We remark that G is a simple connected graph of order $n \geq 3$.

We start by showing (i). Suppose first that $G = K_{1,n-1}$. From the definition of $\mathrm{R}(G)$ it follows that $m = 2(n-1)+1$. Furthermore, if v is the vertex of G with degree $n-1$, then the degree of v in $\mathrm{R}(G)$ is $2(n-1) = m-1$. This last and Proposition 2.9(a) imply that $\partial(\mathrm{R}(G)) = m-2$.

Now suppose that $\partial(\mathrm{R}(G)) = m-2$. Then $\mathrm{R}(G)$ must have a vertex which is adjacent to any other vertex. This and the fact that $n \geq 3$, imply that $v \in V$. In particular, v must be adjacent to any other vertex of G, and so G contains $K_{1,n-1}$ as subgraph. Since v is adjacent to any other vertex of $\mathrm{R}(G)$, then any edge of G must belong to its subgraph $K_{1,n-1}$, and so $G = K_{1,n-1}$.

Now we show (ii). Suppose first that $G = K_{1,n-1} \cup \{e\}$, where e is an edge joining two leaf vertices of $K_{1,n-1}$. Then the apex vertex v of G has degree $m-2$ in $\mathrm{R}(G)$, and by Proposition 2.9(b) we must have that $\partial(\mathrm{R}(G)) = m-3$.

Now assume that $\partial(\mathrm{R}(G)) = m-3$. From $n \geq 3$ and the fact that G is connected we have that $m \geq 5$. On the other hand, note that $\partial(\mathrm{R}(G)) = m-3$ implies that the maximum degree of $\mathrm{R}(G)$ is $m-2$.

Let $v \in V(\mathrm{R}(G))$ with $\delta(v) = m-2 \geq 3$. Since $\delta(u) = 2$ for any $u \in U$, then we have that v must belong to V. Since for each edge of G with both ends in $V \setminus \{v\}$ there exists a unique vertex in U which is not adjacent to v and $\delta(v) = m-2$, then we have that G has at most one edge with both ends in $V \setminus \{v\}$. On the other hand, $\delta(v) = m-2$ implies that $\mathrm{R}(G)$ contains exactly one vertex, say v_1, which is not adjacent to v, and so $C(\{v\}) = \{v_1\}$. From the connectivity of $\mathrm{R}(G)$ we know that $B(\{v\})$ contains a vertex v_2, which is adjacent to v_1.

Note that if $v_2 \in U$, then we must have that v_2 is adjacent to v in $\mathrm{R}(G)$, and hence that v_1 is adjacent to v in G, contradicting that $v_1 \in C(\{v\})$. This implies that any neighbour of v_1 in $\mathrm{R}(G)$ must belong to $V \setminus \{v\}$, and hence that $v_1 \in U$. Let v_3 be the other neighbour of v_1. Then the edge $e = v_2v_3$ belongs to G, and so $G = K_{1,n-1} \cup \{e\}$, as required. □

Proposition 2.11. *Let p and q be positive integers, such that $p+q \geq 4$ and $p < q$. Let $\{P, Q\}$ be the vertex bipartition of $K_{p,q}$, with $|P| = p$ and $|Q| = q$. Then P is the only differential set of* $\mathrm{R}(K_{p,q})$.

Proof. Let $\{P, Q\}$ be the vertex partition of $V(K_{p,q})$ with $|P| = p < q = |Q|$. Since the assertion is easy to verify for $n = 4, 5, 6$, we will assume that $n > 6$.

Let $m := |V(\mathrm{R}(K_{p,q}))| = pq + q + p$. Since P is a vertex cover of $K_{p,q}$, then P is a dominating set of $\mathrm{R}(K_{p,q})$, $\partial_{\mathrm{R}(K_{p,q})}(P) = m - 2p$, and so $\partial(\mathrm{R}(K_{p,q})) \geq m - 2p$.

From $\partial(\mathrm{R}(K_{p,q})) \geq m - 2p$ and Observation 2.8, it follows that the size of any differential set of $\mathrm{R}(K_{p,q})$ is at most p.

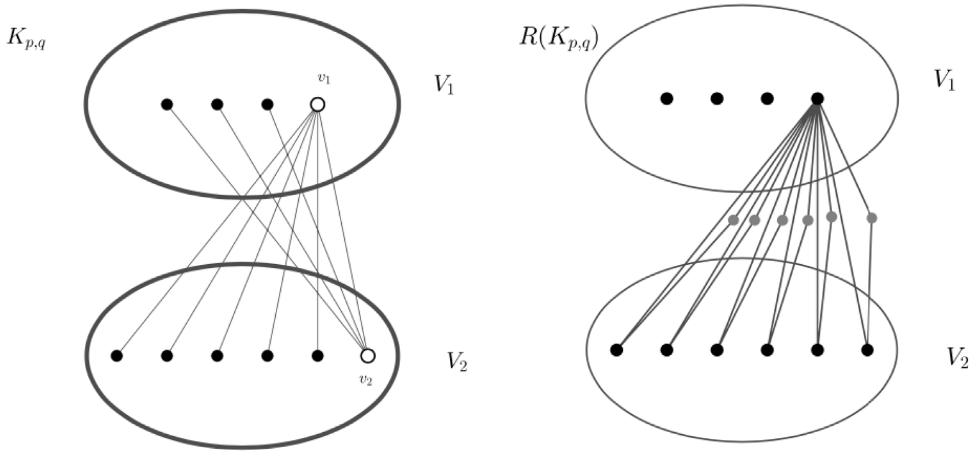

FIGURE 3. The complete bipartite graph $G = K_{p,q}$ shows that the structure of a differential set of G can be very different from the structure of a differential set of $\mathrm{R}(G)$.

Let S be a differential set of $\mathrm{R}(K_{p,q})$, and let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. By Proposition 2.3, we can assume that $S \subseteq V$.

Now we will show that $|S| = p$. Since $|S| \leq p$, it suffices to show that $|S| \geq p$. Seeking a contradiction, suppose that $|S| \leq p-1$. Then each of P and Q has at least a vertex which is not in S. Let $v \in Q \setminus S$, and suppose that $P \setminus S$ has at least three vertices, say v_1, v_2, and v_3. For $i = 1, 2, 3$, let u_i be the only vertex of U which is adjacent to both v and v_i. Note that the existence of such $u_1, u_2, u_3, v_1, v_2, v_3$ and v implies, $\partial_{\mathrm{R}(K_{p,q})}(S \cup \{v\}) = \partial_{\mathrm{R}(K_{p,q})}(S) + 1$, a contradiction. Thus we conclude that P has at most 2 vertices which are not in S. By an analogous reasoning, we can conclude that Q has at most 2 vertices which are not in S. These imply $|S| \geq p + q - 4$, and hence that $p - 1 \geq p + q - 4$. But this last inequality implies, $3 \geq q \geq n/2$, a contradiction.

From $|S| = p$ and the properties of P we can conclude that P is a differential set of $\mathrm{R}(K_{p,q})$. Since $S \subseteq V$ and $V = P \cup Q$, it remains to show that $S \subseteq P$.

Assume by way of contradiction that $S \nsubseteq P$. Since $q > p$, there exists a vertex $v \in Q \setminus S$. If $P \setminus S$ has at least two vertices, then $S \cup \{v\}$ is a differential set of $\mathrm{R}(K_{p,q})$ of size $p+1$, which is impossible. Thus, we can assume that $P \setminus S$ has exactly one vertex, say u. Since $q > 3$, $Q \setminus S$ has at least two vertices and as above, $S \cup \{u\}$ must be a differential set of $\mathrm{R}(K_{p,q})$ of size $p+1$, which is again impossible. □

Note that if $q > p \geq 3$, then any differential set of $K_{p,q}$ consists of exactly two vertices belonging to distinct parts of the vertex bipartition of $V(K_{p,q})$. On the other hand, we have proved in Proposition 2.11 that for the same range of values of p and q, the only differential set of $\mathrm{R}(K_{p,q})$ is the smaller part of $V(K_{p,q})$. These facts show that the structure of a differential set of G can be considerably distinct from the structure of a differential set of $\mathrm{R}(G)$. See Figure 3 for an example.

We recall that the *wheel graph* W_n of order $n \geq 4$ is formed by the cycle C_{n-1}, plus an additional vertex, say v, such that v is adjacent to any vertex in $V(C_{n-1})$. Usually, the vertex v is called the *apex vertex* of W_n.

Proposition 2.12. *Let p, q, n be positive integers, such that $p + q = n \geq 4$ and $p \leq q$. The following equalities hold:*

(i) $\partial(\mathrm{R}(K_n)) = \dfrac{n(n-1)}{2} - n + 3.$
(ii) $\partial(\mathrm{R}(W_n)) = 2n - 3.$
(iii) $\partial(\mathrm{R}(K_{p,q})) = q(p+1) - p.$

Proof. Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$, where G is K_n, W_n or $K_{p,q}$, depending on the case under consideration.

First we show (i). Then $G = K_n$, $|V| = n$ and $|U| = \frac{n(n-1)}{2}$. Let S be a differential set of $\mathrm{R}(G)$. By Proposition 2.3, we can assume that $S \subseteq V$.

We note that if $v \in V$, then v has degree $2(n-1)$ in $\mathrm{R}(G)$, and so $\partial_{\mathrm{R}(G)}(S) \geq \partial_{\mathrm{R}(G)}(\{v\}) = 2n - 3 > 0$. Since $\partial_{\mathrm{R}(G)}(\emptyset) = 0$, then S cannot be the empty set.

We claim that $|S| \geq n - 3$. Suppose not. Then $V \setminus S$ contains at least 4 vertices, say v_1, v_2, v_3 and v_4. Then these 4 vertices must belong to $B_{\mathrm{R}(K_n)}(S)$. For $i = 2, 3, 4$, let u_i be the only vertex of U which is adjacent to v_1 and v_i. Since $v_1, v_2, v_3, v_4 \in B_{\mathrm{R}(K_n)}(S)$, then u_2, u_3 and u_4 are in $C_{\mathrm{R}(K_n)}(S)$. From these facts it follows that $\partial_{\mathrm{R}(K_n)}(S \cup \{v_1\}) > \partial_{\mathrm{R}(K_n)}(S)$, contradicting that S is a differential set of $\mathrm{R}(K_n)$. Thus we must have that $|S| \in \{n-3, n-2, n-1, n\}$.

On the other hand, the following is easy to check:

$$\partial_{\mathrm{R}(K_n)}(S) = \begin{cases} \dfrac{n(n-1)}{2} - n + 3 & \text{if } \ |S| = n-3, n-2, \\[2ex] \dfrac{n(n-1)}{2} - n + 2 & \text{if } \ |S| = n-1, \\[2ex] \dfrac{n(n-1)}{2} - n & \text{if } \ |S| = n. \end{cases} \tag{1}$$

From equation (1) and the fact that these four are the only possibilities for such a set S, we can conclude that $\partial(\mathrm{R}(K_n)) = \dfrac{n(n-1)}{2} - n + 3$, as required.

Now we show (ii). Let v be the apex vertex of W_n. Note that $|V(\mathrm{R}(W_n))| = 3(n-1) + 1 = 3n - 2$ and also that $\partial_{\mathrm{R}(W_n)}(\{v\}) = 2(n-1) - 1$. This last equality implies $\partial(\mathrm{R}(W_n)) \geq 2n - 3$. Since the case in which $n = 4$ is easy to verify, we assume that $n \geq 5$.

Let S be a differential set of $\mathrm{R}(W_n)$, and let $\ell := |S| > 0$. By Proposition 2.3, we can assume that $S \subseteq V$. Moreover, we claim that if $v \notin S$, then $S' := S \cup \{v\}$ is also a differential set of $\mathrm{R}(W_n)$. Indeed, suppose that $v \notin S$, and that $\partial_{\mathrm{R}(W_n)}(S) > \partial_{\mathrm{R}(W_n)}(S')$. Then the last inequality implies that $V(C_{n-1})$ has at most one vertex not in S. This, the supposition that $v \notin S$, and $S \subseteq V$ imply that $|S| \geq n - 2$. Hence $\partial_{\mathrm{R}(W_n)}(S) \leq ((3n-2) - 1) - 2|S| \leq 3n - 3 - 2(n-2) = n + 1$. Since $\partial(\mathrm{R}(S)) \geq 2n - 3$, then we have that $n + 1 \geq 2n - 3$, or that $4 \geq n$. Since $4 \geq n$ contradicts $5 \leq n$, we have that S' is also a differential set of $\mathrm{R}(W_n)$.

Let $v, v_{i_1}, \ldots, v_{i_\ell}$ be the vertices of S'. We claim that $S'' := S' \setminus \{v_{i_\ell}\}$ is also a differential set of $\mathrm{R}(W_n)$. Indeed, since $|\mathrm{epn}[v_{i_\ell}, S']| \leq 2$ and $|S''| = |S'| - 1$, then we must have that $\partial_{\mathrm{R}(W_n)}(S') = \partial_{\mathrm{R}(W_n)}(S'')$, as required. Similarly, it can be deduced that $S''' := S'' \setminus \{v_{i_{\ell-1}}\}$ is also a differential set of $\mathrm{R}(W_n)$. Continuing in this way, we can conclude that $\{v\}$ is a differential set of W_n, and hence that $\partial(\mathrm{R}(W_n)) = \partial_{\mathrm{R}(W_n)}(\{v\}) = 2n - 3$, as required.

By Proposition 2.11, we have (iii). □

The following proposition shows a surprising relationships between the vertex cover number of G and the domination number of $\mathrm{R}(G)$.

Proposition 2.13. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. Then $\tau(G) = \gamma(\mathrm{R}(G))$.*

Proof. First we show that $\gamma(\mathrm{R}(G)) \leq \tau(G)$. Let $X \subseteq V(G)$ be a vertex cover of G such that $\tau(G) = |X|$. It suffices to show that X is a dominating set of $\mathrm{R}(G)$. Let u be a vertex of $V(\mathrm{R}(G))$. Then we need show that $u \in X$ or that $\mathrm{R}(G)$ contains an edge ux with $x \in X$. If $u \in X$ there is nothing to show. Then we can assume that $u \notin X$. If $u \in U$, then we know that u has exactly two neighbours v_1 and v_2 such that $v_1, v_2 \in V$ and $v_1v_2 \in E(G)$. Since X is a vertex cover of G, then at least one of v_1 or v_2 must belong to X, and hence such a vertex is the required x. Thus we can assume that $u \in V$. Since G is a connected graph of order $n \geq 2$, then u has at least one neighbour in V, say y. From the fact that X is a vertex cover of G, and the fact $u \notin X$ we have that y must belong to X, and so y is the required x.

Now we show that $\gamma(\mathrm{R}(G)) \geq \tau(G)$. From Proposition 2.2, we know that V contains a minimum dominating set S of $\mathrm{R}(G)$. Then it is suffices to show that S is a vertex cover of G. Let v_1v_2 be an edge of G. From the definition of $\mathrm{R}(G)$ we know that U contains a unique vertex u such that $N(u) = \{v_1, v_2\}$. Since such a vertex u must be dominated by some vertex of S, then at least one of v_1 or v_2 is in S, and hence S is a vertex cover of G. This implies that $\tau(G) \leq \gamma(\mathrm{R}(G))$. □

Proposition 2.14. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. If $S \subseteq V$ is a vertex cover of G and $\partial(G) = \partial(S)$, then S is a differential set of $\mathrm{R}(G)$.*

Proof. Let S be as in the hypothesis of Proposition 2.14. Since S is a vertex cover of G, then S is a dominating set for both G and $\mathrm{R}(G)$. We note that if D is any dominating set of G, then $|V| - 2|D| = \partial(D) \leq \partial(G) = \partial(S) = |V| - 2|S|$, and hence S is a minimum dominating set of G. On the other hand, since G is an induced subgraph of $\mathrm{R}(G)$, then $|S| = \gamma(G) \leq \gamma(\mathrm{R}(G))$. From the last inequality and the fact that S is a dominating set of $\mathrm{R}(G)$, we have that S is also a minimum dominating set of $\mathrm{R}(G)$. Thus, $\gamma(\mathrm{R}(G)) = |S|$.

Let $X \subseteq V(\mathrm{R}(G))$ be a differential set of $\mathrm{R}(G)$, and let $m := |V(\mathrm{R}(G))|$. Since S is a dominating set also for $\mathrm{R}(G)$, then $m - 2|S| = \partial(S) \leq \partial(X) \leq m - 2|X|$, and so $|S| \geq |X|$. By Corollary 2.5, we can assume that X is a dominating set of G and that $X \subseteq V$. Since $S \subseteq V$ is a minimum dominating set of G, then $|X| \geq |S|$, and so $|X| = |S| = \gamma(\mathrm{R}(G))$.

Since $X \cup C(X)$ is a dominating set of $\mathrm{R}(G)$, then $\gamma(\mathrm{R}(G)) \leq |X \cup C(X)| = |X| + |C(X)| = m - |B(X)|$. Thus $|B(X)| \leq m - \gamma(\mathrm{R}(G))$, and so

$$\partial(\mathrm{R}(G)) = \partial(X) = |B(X)| - |X| \leq \{m - \gamma(\mathrm{R}(G))\} - |X| = m - 2|S| = \partial(S) \leq \partial(\mathrm{R}(G)).$$

□

We note that if S is a differential set of G and $\mathrm{R}(G)$, then S is not necessarily a vertex cover of G. See for instance the graph in Figure 4.

3. General tight bounds for $\partial(\mathrm{R}(G))$

Our goal in this section is to show Theorem 3.3, which states general lower and upper bounds for $\partial(\mathrm{R}(G))$.

We remark that a subset $S \subseteq V(G)$ is $k-$*dependent* in G if $\langle S \rangle$ has maximum degree at most k. For more about k-dependent sets we refer the reader to [26].

Proposition 3.1. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. If $S \subseteq V$ is a differential set of $\mathrm{R}(G)$, then $\langle B_G(S) \rangle$ is 2-dependent. Moreover, if S is a maximal differential set of $\mathrm{R}(G)$, then $\langle B_G(S) \rangle$ is 1-dependent.*

Proof. We recall that $|V(G)| = n \geq 3$. Let $S \subseteq V$ be a differential set of $\mathrm{R}(G)$. Seeking a contradiction, suppose that $\langle B_G(S) \rangle$ has a vertex v such that $\delta_{\langle B_G(S) \rangle}(v) \geq 3$. Let v_1, v_2 and v_3 be three distinct neighbours of v in $\langle B_G(S) \rangle$, and for $i = 1, 2, 3$, let u_i be the only vertex in U that is adjacent to both v and v_i. Then $u_1, u_2, u_3 \in C_{\mathrm{R}(G)}(S) \cap N_{\mathrm{R}(G)}(v)$, and hence we have $\partial_{\mathrm{R}(G)}(S \cup \{v\}) \geq \partial_{\mathrm{R}(G)}(S) + 1$, a contradiction.

Now suppose that, additionally, S is maximal and that has $\langle B_G(S) \rangle$ a vertex v of degree 2. Let v_1 and v_2 be two distinct neighbours of v. For $i = 1, 2$, let u_i be the only vertex in U that is adjacent to both v and v_i. Then $u_1, u_2 \in C_{\mathrm{R}(G)}(S) \cap N_{\mathrm{R}(G)}(v)$, and hence we have $\partial_{\mathrm{R}(G)}(S \cup \{v\}) \geq \partial_{\mathrm{R}(G)}(S)$, which contradicts the maximality of S. □

If $S \subseteq V(G)$ is a differential set of $\mathrm{R}(G)$ of maximum cardinality, then we shall use $\mu(G)$ to denote $|S|$. Similarly, we shall use $\lambda(G)$ to denote $|E(G)| - |V(G)| + 2\alpha(G)$. Note that the connectivity of G and $\alpha(G) \geq 1$ imply that $\lambda(G) \geq 1$.

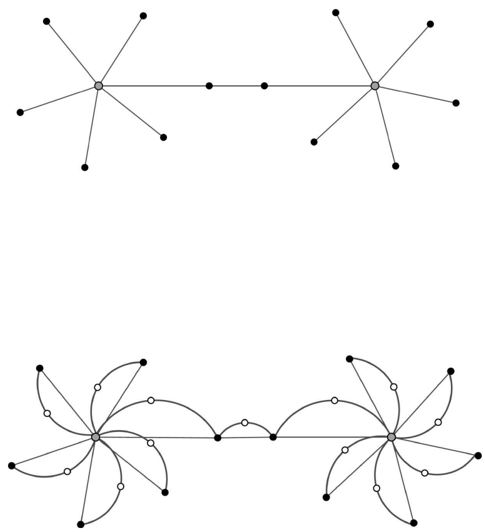

FIGURE 4. The gray vertices form a differential set of G and $\mathrm{R}(G)$, but they are not a vertex cover of G.

Proposition 3.2. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. If $S \subseteq V$ is a maximum differential set of $\mathrm{R}(G)$, then $|C_{\mathrm{R}(G)}(S)| \leq \frac{|V(G)|-\mu(G)}{2}$.*

Proof. Let $S \subseteq V$ be a maximum differential set of $\mathrm{R}(G)$, and let H be the subgraph of G induced by $B_G(S)$. We claim that S is a dominating set in G. Suppose not, and let v be a vertex in $V \cap C_{\mathrm{R}(G)}(S)$. Since G is connected and $|V(G)| = n \geq 2$, then v is adjacent to another vertex $v_1 \in V$.

Let u_1 be the only vertex of U which is adjacent to both v and v_1. Since S is a maximum differential set of $\mathrm{R}(G)$ and $v \in C_{\mathrm{R}(G)}(S)$, then we must have that $v_1 \in B_{\mathrm{R}(G)}(S)$ and that $u_1 \in C_{\mathrm{R}(G)}(S)$. These last two conclusions imply that $S \cup \{v\}$ is also a differential set of $\mathrm{R}(G)$, contradicting the maximality of S. Then, we can assume that $C_{\mathrm{R}(G)}(S) \subseteq U$. In particular, this implies that each $u \in C_{\mathrm{R}(G)}(S)$ corresponds to an edge of H, and hence $|C_{\mathrm{R}(G)}(S)|$ is exactly the number of edges of H. On the other hand, since H has maximum degree at most 1 by Proposition 3.1, then H has at most $\frac{|V(G)|-\mu(G)}{2}$ edges, as required. $\square$

Theorem 3.3. *Let $\{V, U\}$ be the canonical partition of $V(\mathrm{R}(G))$. Then $\lambda(G) \leq \partial(\mathrm{R}(G)) \leq \lambda(G) + \lfloor \frac{|V(G)|-\mu(G)}{2} \rfloor$.*

Proof. We recall that $|V(G)| = n \geq 2$. First we shall show $\lambda(G) \leq \partial(\mathrm{R}(G))$. Let I be a maximum independent set of G, and let $S := V \setminus I$. Then $\alpha(G) = |I|$ and $|S| = n - \alpha(G)$. Note that $B_{\mathrm{R}(G)}(S) = U \cup I$. Then $\partial(\mathrm{R}(G)) \geq \partial_{\mathrm{R}(G)}(S) = |U| + |I| - |S| = |E(G)| + \alpha(G) - (n - \alpha(G)) = |E(G)| + 2\alpha(G) - n = \lambda(G)$, as required.

Now we shall show that $\partial(\mathrm{R}(G)) \leq \lambda(G) + \lfloor \frac{n-\mu(G)}{2} \rfloor$. Let S be a maximum differential set of $\mathrm{R}(G)$. Then $\mu(G) = |S|$. Moreover, by Proposition 2.3 we can assume that $S \subseteq V$. Also note that the maximality of S implies that $C_{\mathrm{R}(G)}(S) \subseteq U$. Thus V has exactly $n - \mu(G)$ vertices in $B_{\mathrm{R}(G)}(S)$. Let F be the set of edges of G which have both end vertices in $V \setminus S$ and let $f := |F|$. From Proposition 3.2 we know that $f \leq \lfloor \frac{n-\mu(G)}{2} \rfloor$, and that F is a matching of G. Let S' be a set of vertices in V which results by adding to S exactly one vertex of each edge in F. Then S' is a dominating set of $\mathrm{R}(G)$, and $I' := V \setminus S'$ is an independent set of G. Clearly, $\lambda(G) = |E(G)| + \alpha(G) - (n - \alpha(G)) \geq |E(G)| + |I'| - (n - |I'|) = \partial_{\mathrm{R}(G)}(I') = \partial(\mathrm{R}(G)) - f$. Then $\lambda(G) \geq \partial(\mathrm{R}(G)) - f \geq \partial(\mathrm{R}(G)) - \lfloor \frac{n-\mu(G)}{2} \rfloor$, as required. $\square$

For r a positive integer, let us denote by $K'_{r,2r}$ the graph that results from the complete bipartite graph $K_{r,2r}$ when we add a matching of r edges with both end vertices in the bigger part of the bipartition of $V(K_{r,2r})$.

Our next result shows that both bounds in Theorem 3.3 are tight.

Proposition 3.4. *Let $r \geq 2$ be a positive integer. Then the following hold:*

(i) $\partial(\mathrm{R}(K_{r,2r})) = \lambda(K_{r,2r})$.

(ii) $\partial(\mathrm{R}(K'_{r,2r})) = \lambda(K'_{r,2r}) + \left\lfloor \frac{3r-\mu(K'_{r,2r})}{2} \right\rfloor$.

Proof. As usual, throughout this proof we assume that $\{V, U\}$ is the canonical partition of $V(\mathrm{R}(G))$, where G is $K_{r,2r}$ or $K'_{r,2r}$ depending on the case under consideration.

First we show (i). By Proposition 2.12(iii) we have $\partial(\mathrm{R}(K_{r,2r})) = 2r(r+1) - r = 2r^2 + r$, and through a direct calculation, we obtain $\lambda(K_{r,2r}) = 2r(r) - 3r + 4r = 2r^2 + r$. Then $\partial(\mathrm{R}(K_{r,2r})) = \lambda(K_{r,2r})$, as claimed.

Now we shall show (ii). For brevity, we will use G to denote $K'_{r,2r}$. Without loss of generality, we assume that

– $P := \{v_1, \dots, v_r\}, Q := \{w_1, \dots, w_{2r}\}$, and
– $E(G) := \{v_i w_j | v_i \in P \text{ and } w_j \in Q\} \cup \{w_i w_{i+r} | i = 1, \dots, r\}$.

Let S be a maximum differential set of $\mathrm{R}(G)$. Then $\mu(G) = |S|$ and $\partial(G) = \partial_{\mathrm{R}}(G)(S)$. As before, by Proposition 2.3 we can assume that $S \subseteq V = P \cup Q$. It is a routine exercise to show that the domination number of $\mathrm{R}(G)$ is exactly $2r$. In view of this, we can assume that $|S| \leq 2r$.

Each of the following equalities follows directly from the definitions involved: (i) $\alpha(G) = r$, (ii) $\lambda(G) = (2r(r) + r) - 3r + 2r = 2r^2$, and (iii) $\partial_{\mathrm{R}(G)}(P) = 2r(r) + |Q| - |P| = 2r(r) + 2r - r = 2r^2 + r$.

In particular, note that if $S = P$, then there is nothing to prove. Indeed, in such a case we have that $\mu(G) = r$, and hence $\partial_{\mathrm{R}(G)}(P) = 2r^2 + r = 2r^2 + \lfloor \frac{3r-r}{2} \rfloor$, as required. Thus, from now on we assume that $S \neq P$. Let $P_1 := P \cap S, Q_1 := Q \cap S, P_0 := P \setminus P_1$, and $Q_0 := Q \setminus Q_1$.

Claim 1. $|Q_0| \geq 1$. Seeking a contradiction, suppose that $Q_0 = \emptyset$. Then $Q \subseteq S$, and hence we must have that $Q = S$. This last implies that $\partial_{\mathrm{R}(G)}(S) = |U| + |P| - |S| = (2r(r) + r) + (r) - (2r) = 2r^2 < 2r^2 + r = \partial_{\mathrm{R}(G)}(P)$, which contradicts the choice of S.

Claim 2. $|Q_1| \geq 1$. Seeking a contradiction, suppose that $Q_1 = \emptyset$. Since $S \neq P$, then $P_0 \neq \emptyset$, and hence $|S| < |P| = r$. Then $\partial_{\mathrm{R}(G)}(S) = (2r)|S| + |Q| - |S| = |S|(2r-1) + 2r < r(2r-1) + 2r = 2r^2 + r = \partial_{\mathrm{R}(G)}(P)$, contradicting the choice of S.

Claim 3. $|P_0| \geq 1$. Seeking a contradiction, suppose that $P_0 = \emptyset$. This supposition and Claim 2 imply that $S = P \cup Q_1$ where $Q_1 \neq \emptyset$. Let U_1 be the set of vertices of U which are adjacent to some vertex of Q_1 but not to a vertex of P. From the definition of G and $\mathrm{R}(G)$ it is clear that $|Q_1| \geq |U_1|$. Thus $\partial_{\mathrm{R}(G)}(S) = (2r(r) + |Q| - |Q_1| + |U_1|) - (|P| + |Q_1|) = (2r^2 + |Q| - |P|) + (|U_1| - 2|Q_1|) = \partial_{\mathrm{R}(G)}(P) + |U_1| - 2|Q_1|$. Since $|U_1| - 2|Q_1| < 0$, then $\partial_{\mathrm{R}(G)}(S) < \partial_{\mathrm{R}(G)}(P)$, which contradicts the choice of S.

Claim 4. $|P_1| \geq 1$. Seeking a contradiction, suppose that $P_1 = \emptyset$, and hence that $P = P_0$. From Claim 1 we know that Q_0 contains at least one vertex, say w. Let U_w be the set of vertices of U which have a neighbour in P and the other in $\{w\}$. Since $|P| = r \geq 2$, then $|U_w| \geq 2$. Since no vertex in U_w belongs to $B_{\mathrm{R}(G)}(S)$, then for $S' := S \cup \{w\}$ we have that $\partial_{\mathrm{R}(G)}(S') \geq \partial_{\mathrm{R}(G)}(S)$, contradicting the choice of S.

Claim 5. $|Q_0| = 1$. Seeking a contradiction and considering Claim 1, we can suppose that $|Q_0| \geq 2$. From Claim 3 we know that P_0 contains at least one vertex, say v. Let U_v be the set of vertices of U which have a neighbour in $\{v\}$ and the other in Q_0. Since $|Q_0| \geq 2$, then $|U_v| \geq 2$. Since no vertex in U_v belongs to $B_{\mathrm{R}(G)}(S)$, then for $S' := S \cup \{v\}$ we have that $\partial_{\mathrm{R}(G)}(S') \geq \partial_{\mathrm{R}(G)}(S)$, contradicting the choice of S.

Claim 5 implies that $|Q_1| = 2r - 1$. This, together with $|S| \leq 2r$ and Claim 4 imply that $|P_1| = 1$, and hence that $|S| = 2r$. Since $|Q_0| \geq 1$ and $|P_0| \geq 1$, then the set $U_{0,0}$ of vertices of U which have a neighbour in Q_0 and the other in P_0 is nonempty. Moreover, since no vertex in $U_{0,0}$ belongs to $B_{\mathrm{R}(G)}(S)$, then S is not a dominating set of $\mathrm{R}(G)$, but it has cardinality $2r$. Thus $\partial_{\mathrm{R}(G)}(Q) > \partial_{\mathrm{R}(G)}(S)$, contradicting the choice of S. $\square$

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